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Eigenfunction Expansions Associated
with the Laplacian for Certain
Domains with Infinite Boundaries

Charles Irwin Goldstein

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ABSTRACT

The purpose of this dissertation is to prove an expansion theorem for the negative Laplacian $-\Delta$ acting on functions with zero boundary conditions in a perturbed semi-infinite cylinder in N -dimensional Euclidean space ($N \geq 2$).

The case of an unperturbed cylinder is discussed first. The spectral multiplicity is completely determined and a complete, orthogonal set of general eigenfunction is explicitly given.

The finite end of the cylinder is then subjected to certain perturbations. Two sets of generalized eigenfunctions are constructed for the perturbed cylinder. They are constructed using the method of limiting absorption. The behavior of these eigenfunctions at infinity is described by means of a radiation condition analogous to the Sommerfeld radiation condition for the exterior problem.

These functions are shown to form a complete, orthogonal set of generalized eigenfunctions for certain types of perturbed cylinders. Finally, it is shown for a wider class of domains that there can be at most a finite number of linearly independent generalized eigenfunctions associated with each real number.

INTRODUCTION

In this paper we shall prove an expansion theorem for the self-adjoint operator A_Ω given by the Friedrichs extension [5] of the operator $-\Delta$ acting on $C_0^\infty(\Omega)$, where Ω is a perturbed semi-infinite cylinder in N -dimensional Euclidean space, R_N ($N \geq 2$). We shall construct two complete sets of generalized eigenfunctions for the operator A_Ω , describe the behavior of these functions at infinity and determine the spectral multiplicity $m(\lambda)$ at each point λ in the spectrum of A_Ω .

The domain Ω consists of two parts, Ω_1 and Ω_2 . Ω_1 is finite (i.e. contained in some sphere) and Ω_2 is a semi-infinite cylinder with arbitrary cross section. We impose the additional condition on Ω that the outward normal at every point of the boundary make an angle of at least 90° with a fixed direction. Rellich [12] has proven that in this case, A_Ω has no point eigenvalues. (At the end of Chapter 3 and in the appendix we shall consider more general cases in which eigenvalues are present.)

The generalized eigenfunctions to be used in the proof of the expansion theorem will be referred to as distorted plane waves. Distorted plane waves were first used for

proving an expansion theorem by Ikebe [7]. He treated the case of the Schroedinger operator $-\Delta + q(x)$ acting on $L_2(\mathbb{R}^3)$, $q(x)$ being a potential function. Ikebe used classical distorted plane waves, $w(x;\xi) = e^{-ix \cdot \xi} + v(x;\xi)$ where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, and $x \cdot \xi = \sum_{i=1}^3 x_i \xi_i$.

The plane waves $e^{-ix \cdot \xi}$ are generalized eigenfunctions for the operator $-\Delta$ on $L_2(\mathbb{R}^3)$. The outgoing (incoming) waves $v(x;\xi)$ satisfy the Sommerfield radiation condition at infinity. These distorted plane waves were also used by Shenk [14] and Shizuta [15] in dealing with the operator $-\Delta$ acting on $L_2(D)$, D being the exterior of a bounded domain with smooth boundary. Shenk proved that for $D \subset \mathbb{R}^N$, this operator is unitarily equivalent to the operator $-\Delta$ acting on $L_2(\mathbb{R}^N)$ ($N \geq 2$). This result was also proven in a paper by Lax and Phillips, [10], (for N odd), employing a different method.

In Chapter 1 we state an expansion theorem for the case of an unperturbed semi-infinite cylinder S . The generalized eigenfunctions for the operator A_S will be called plane waves. The simplest example of this type of domain is the two-dimensional strip S_2 : $0 \leq x_1$, $0 \leq x_2 \leq \pi$. The plane waves $\sin \xi x_1 \sin n x_2$ ($\xi \in [0, \infty)$, $n = 1, 2, \dots$) form a complete set of generalized eigenfunctions. Furthermore, for each λ such that $n^2 \leq \lambda < (n+1)^2$, we have $m(\lambda) = n$. The spectrum of A_{S_2} is the infinite

interval $[1, \infty)$. As will be seen in Chapter 1, analogous results hold for a cylinder of arbitrary cross section in R^N .

In Chapter 2 we construct the generalized eigenfunctions for the operator A_Ω . These functions, to be denoted by $w_n(x; \xi)$, will be expressed as the sum of a plane wave and an outgoing (incoming) wave. The outgoing (incoming) waves, $v_n(x; \xi)$, satisfy a radiation condition at infinity different from the Sommerfield radiation condition. This condition involves the form of the Fourier coefficients of $v_n(x; \xi)$. For example, suppose $\Omega \subset R^2$, and $\Omega_2 = S_2$. In this case $v_n(x; \xi)$ satisfies the boundary value problem: $(-\Delta - \xi^2 - n^2)v_n(x; \xi) = 0$, $v_n(x; \xi) = -\sin \xi x_1 \sin n x_2$ on the boundary of Ω . In addition, $v_n(x; \xi)$ satisfies the following radiation condition for x_1 sufficiently large:

$$\int_0^\pi v_n(x; \xi) \sin j x_2 \, dx_2 = c_j e^{-\sqrt{j^2 - (\xi^2 + n^2)} x_1},$$

$$j = 1, 2, \dots$$

The problem of constructing the functions $w_n(x; \xi)$ is first reduced to that of solving the boundary value problem:

$$(-\Delta - \lambda)\mu = F,$$

$\mu = 0$ on the boundary of Ω , where $\lambda > 0$, $F(x) \in C^\infty(\Omega)$, and $F(x)$ vanishes for $|x|$ large. This boundary value problem is then solved using the principle of limiting absorption [4]. It is shown that as $\varepsilon \downarrow 0$ ($\varepsilon \uparrow 0$), the functions $\mu_\varepsilon = (A - \lambda - i\varepsilon)^{-1}F$ converge in a certain sense to the unique solution of the boundary value problem, which satisfies the radiation conditions.

In Chapter 3 we prove that the distorted plane waves $w_n(x; \xi)$ form a complete, orthogonal set of generalized eigenfunctions for the operator A_Ω . The completeness of the generalized eigenfunctions $w_n(x; \xi)$ follows from Theorem 3.5. The orthogonality relation is then established in Theorem 3.6. This is proven employing the method used by Ikebe in [6]. Finally the results are extended to include certain cases for which point eigenvalues are present.

In the appendix we consider a more general class of domains. The theorem proven implies that there can be at most a finite number of linearly independent generalized eigenfunctions associated with any real number λ . The proof is based on a generalized Phragmén-Lindelöf theorem due to Lax [9], dealing with the solutions of elliptic boundary value problems in a perturbed semi-infinite cylinder.

CHAPTER 1 UNPERTURBED EXPANSION THEOREM

In this chapter we state some preliminary results. The most important result of the chapter is an expansion theorem, analogous to the well-known Fourier integral theorem.

Definition 1.1. Suppose B is an arbitrary domain in N -dimensional Euclidean space and $\overset{\circ}{B}$ is its boundary. Set $x = (x_1, \dots, x_N)$, $\tilde{x} = (x_1, \dots, x_{N-1})$. Thus any point in B may be represented as $(\tilde{x}, x_N)$. We denote by A_B the operator $-\Delta$ acting on those functions defined in B which are zero on $\overset{\circ}{B}$. To be more precise let $\overset{\circ}{H}_1(B)$ denote the closure of $C_0^\infty(B)$ under the H_1 norm ($\|\cdot\|_{1(B)}$), where for each function $\psi(x)$ in $C_0^\infty(B)$, we have

$$\|\psi\|_{1(B)}^2 = \int_B |\psi(x)|^2 dx + \sum_{j=1}^N \int_B \left| \frac{\partial \psi(x)}{\partial x_j} \right|^2 dx.$$

The domain of the operator A_B , to be denoted by $D(A_B)$, consists of all functions $\mu(x)$ in $\overset{\circ}{H}_1(B)$ for which there exists a function $w(x)$ in $L_2(B)$ satisfying the condition that

$$\sum_{j=1}^N \int_B \frac{\partial \phi}{\partial x_j} \overline{\frac{\partial \mu}{\partial x_j}} dx = \int_B \phi \bar{w} dx \quad \text{for all } \phi \text{ in } \overset{\circ}{H}_1(B).$$

For each such μ set $A_B \mu = w$.

In this section the domain under consideration will be a semi-infinite cylinder.

Definition 1.2. Denote by S the semi-infinite cylinder $x_N \geq 0$, $\tilde{x} \in \ell$ where ℓ is a finite $N-1$ dimensional domain lying in the hyperplane $x_N = 0$. Denote A_S by A_0 .

We shall state an expansion theorem for functions square-integrable over S . This theorem will then be used to analyze the operator A_0 . First, however, it is necessary to make one more definition.

Definition 1.3. Consider the set of elements of the form $f = \{f_n(\xi)\} = (f_1(\xi), f_2(\xi), \dots)$, where each function $f_n(\xi)$ is square-integrable over $(0, \infty)$ and

$$\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(\xi)|^2 d\xi < \infty.$$

We define an inner product for this set in the usual way.

For any two elements $f = \{f_n(\xi)\}$, $g = \{g_n(\xi)\}$, set

$$(f, g)_H = \sum_{n=1}^{\infty} \int_0^{\infty} f_n(\xi) \overline{g_n(\xi)} d\xi.$$

Denote the resulting Hilbert space by H .

We next construct an isometry between the Hilbert spaces $L_2(S)$ and H . Since ℓ is a bounded domain in $N-1$ dimensional space, it follows that there exists a complete, countable, orthonormal set of eigenfunctions for the operator A_ℓ . Denote the eigenvalues, ordered increasingly, by $\{v_n\}$ and the corresponding orthonormal eigenfunctions by $\eta_n(\tilde{x})$. It is well known that $v_n \geq 0$ and $v_n \uparrow \infty$ as $n \uparrow \infty$. We now define a linear mapping T_0 from $L_2(S)$ into H as follows: For each $f(x)$ in $L_2(S)$, set

$$T_0 f = \lim_{X_N \rightarrow \infty} \left\{ \left(\frac{2}{\pi} \right)^{1/2} \int_{\ell} \int_0^{X_N} \frac{\sin \xi x \eta_n(\tilde{x})}{\sin \xi x} \cdot f(\tilde{x}, x_N) dx_N d\tilde{x} \right\}$$

where $\lim$ means that the limit is to be taken in the sense of the Hilbert space H . Denote $T_0 f$ by $\hat{f}^0 = \{\hat{f}_n^0(\xi)\}$.

We also construct a linear mapping T_1 from H into $L_2(S)$. For each element $g = \{g_n(\xi)\}$ in H , set

$$T_1 g = \left(\frac{2}{\pi} \right)^{1/2} \lim_{\substack{M \rightarrow \infty \\ \bar{\xi} \rightarrow \infty}} \sum_{n=1}^M \int_0^{\bar{\xi}} g_n(\xi) \sin \xi x_N \eta_n(\tilde{x}) d\xi$$

where the limit is to be taken in the sense of $L_2(S)$.

We now state without proof the fundamental result of this chapter. It may be proven in the same way as the ordinary Fourier expansion theorem for functions in $L_2(\mathbb{R}^N)$.

Theorem 1.1. T_0 is a unitary transformation from $L_2(S)$ onto H . $T_1 = T_0^* = T_0^{-1}$.

The functions $w_n^0(x; \xi) = \sin \xi x_N \eta_n(\tilde{x})$ are a set of generalized eigenfunctions associated with the operator A_0 . $w_n^0(x; \xi)$ satisfies the equation $(-\Delta - \xi^2 - v_n)w_n^0(x; \xi) = 0$, as well as the boundary conditions $w_n^0(x; \xi) = 0$ on $\overset{\circ}{S}$. We shall refer to these functions as plane waves. Theorem 1.1 tells us that they form a complete set of generalized eigenfunctions. For each function $\phi(x)$ in $C_0^\infty(S)$, it follows from integration by parts that $T_0(-\Delta\phi) = \{(\xi^2 + v_n)\hat{\phi}_n^0(\xi)\}$. For each n , make the substitution $\lambda = \xi^2 + v_n$. Set $g_n(\lambda) = \hat{f}_n^0(\xi)$ for every function $f(x) \in L_2(S)$. It follows from Theorem 1.1 that

$$\|f\|_{L_2(S)}^2 = \|\hat{f}^0\|_H^2 = \sum_{n=1}^{\infty} \int_{v_n}^{\infty} |g_n(\lambda)|^2 \frac{d\lambda}{2\sqrt{\lambda - v_n}}.$$

Set $H' = \bigoplus_{n=1}^{\infty} L_2(v_n, \infty; \frac{d\lambda}{2\sqrt{\lambda - v_n}})$. We thus conclude that the

mapping which takes $f(x)$ into $\{g_n(\lambda)\}$ is a unitary transformation of $L_2(S)$ onto H' . Denote this mapping by U_0 . Suppose $f \in D(A_0)$ and $U_0 f = \{g_n(\lambda)\}$. Then $U_0(A_0 f) = \{\lambda g_n(\lambda)\}$. Thus U_0 furnishes us with a spectral represen-

tation of the operator A_0 . Note that $g_n(\lambda)$ is defined on (v_n, ∞) . The spectrum of A_0 is continuous and the spectral multiplicity $m(\lambda)$ has the value n for $v_n \leq \lambda < v_{n+1}$. Thus $m(\lambda) < \infty$ and $m(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$.

Our aim in the remainder of this paper is to prove that the operator A_Ω is unitarily equivalent to A_0 whenever Ω belongs to a class of unbounded domains differing little from S . A rigorous definition of the domain Ω will be given in Chapter 2.

CHAPTER 2 CONSTRUCTION OF THE GENERALIZED EIGENFUNCTIONS

In this chapter we give a precise definition of the domain Ω to be considered. We then construct a set of generalized eigenfunctions for the operator A_Ω and prove that these eigenfunctions have certain properties that will be important for the proof of the expansion theorem.

Definition 2.1. Let Ω be a domain in N dimensional space composed of two parts Ω_1 and Ω_2 . Suppose that Ω satisfies the following conditions (see figure 1 for a description of Ω in two dimensional space):

- (1) Ω is contained in the semi-infinite cylinder S described in Definition 1.2.
- (2) Ω_1 is bounded and is contained in $x_N \leq h$, where $0 < h$.
- (3) Ω_2 is the semi-infinite cylinder $x_N \geq h$, $\tilde{x} \in \ell$ where ℓ is the cross section of S . Thus Ω coincides with S for $x_N \geq h$.
- (4) $\overset{\circ}{\Omega}$ (the boundary of Ω) is a C^∞ surface.
- (5) $\cos \phi(s) \leq 0$ for each point $x(s)$ on $\overset{\circ}{\Omega}$ where $\phi(s)$ is the angle between the exterior normal to Ω and the positive x_N -axis.

Definition 2.2. Let $\Omega_{\tilde{x}_N}(\tilde{\Omega}'_{\tilde{x}_N})$ be that part of $\Omega(\tilde{\Omega})$ for which $x_N \leq \tilde{x}_N$. Let $\Omega_{x_N^1 x_N^2}(\tilde{\Omega}'_{x_N^1 x_N^2})$ be that part of $\Omega(\tilde{\Omega})$ for which $x_N^1 \leq x_N \leq x_N^2$. Let $\ell_{\tilde{x}_N}$ represent the intersection of Ω with the hyperplane $x_N = \tilde{x}_N$. Note that $\tilde{\Omega}_{\tilde{x}_N} = \tilde{\Omega}'_{\tilde{x}_N} \cup \ell_{\tilde{x}_N}$.

Denote the operator A_{Ω} by A . Rellich has proven [12] that A has no eigenvalues. It has been shown by D.S. Jones in [8] that the spectrum of A fills the entire infinite interval $[v_1, \infty)$, where we recall that v_n and $\eta_n(\tilde{x})$ represent the eigenvalues and corresponding eigenfunctions of the operator A_{ℓ} . Our aim is to prove that A is unitarily equivalent to $A_{\tilde{\Omega}}$.

Our starting point is the following collection of results due to Eidus, [4], dealing with the resolvent of A and the solution of the boundary value problem $(-\Delta - \lambda)\mu = F$, $\mu = 0$ on $\tilde{\Omega}$, where F is a $C^\infty(\Omega)$ function which vanishes for $x_N \geq h$, and λ is real.

Lemma 2.1. Suppose $\mu(x) = (A - \lambda - i\varepsilon)^{-1}F(x)$, where $\varepsilon > 0$, λ real. Then

$$\int_{\ell_{x_N}} \mu(\tilde{x}, x_N) \eta_j(\tilde{x}) d\tilde{x} = C_j e^{-\sqrt{v_j - \lambda - i\varepsilon} x_N},$$

for each $j = 1, 2, \dots$. That value of the square root is

chosen for which $\operatorname{Re} \sqrt{\nu_j - \lambda - i\varepsilon} > 0$. The C_j are constants.

To prove the lemma, we observe that: (1) the functions $d_j(x_N) = \int_{\ell_{x_N}} \mu(\tilde{x}, x_N) \eta_j(\tilde{x}) d\tilde{x}$ satisfy the differential

equation $d''(x_N) + (\nu_j - \lambda - i\varepsilon)d_j(x_N) = 0$, and

(2) $\mu(x) \in L_2(\Omega)$. Lemma 2.1 immediately follows from this.

For $\lambda \geq \nu_1$, the resolvent $R_\lambda = (A - \lambda)^{-1}$ does not exist as a continuous linear operator acting on the Hilbert space $L_2(\Omega)$, since λ belongs to the spectrum of A . Hence we cannot always solve the boundary value problem $(-\Delta - \lambda)\mu = F$, $\mu = 0$ on $\overset{\circ}{\Omega}$ with solutions $\mu(x) \in L_2(\Omega)$. However, we shall construct a new topological space, to be denoted by χ , in which this boundary value problem will be well posed. χ will consist of those functions defined in Ω which have locally L_2 derivatives up to the second order and are zero on $\overset{\circ}{\Omega}$. We now give a more precise definition.

Definition 2.2. (1) Consider the set of functions $f(x)$ defined in Ω which satisfy the following conditions:
(a) $f \in H_2^{loc}(\Omega)$ (Given any non-negative integer m , the statement $f \in H_m^{loc}(\Omega)$ means that for each interior compact subregion $\Omega' \subset \Omega$, we have $f \in H_m(\Omega')$, i.e. f is square-integrable and has square-integrable derivatives up to order m over Ω').

(b) $\exists$ a sequence of functions $\{f_n(x)\}$ $\ni$: $f_n \in H_1^{loc}(\Omega)$, $f_n(x)$ vanishes in boundary strips along $\overset{\circ}{\Omega}$, and in any region $\Omega_{\overset{\circ}{x}_N}$ we have:

$$\|f_n - f\|_{1(\Omega_{\overset{\circ}{x}_N})} \rightarrow 0.$$

(For any domain B and any non-negative integer m the expression $\|g\|_{m(B)}$ denotes

$$\left(\sum_{|\alpha| \leq m} \int_B |D^\alpha g|^2 dx \right)^{1/2}$$

where we have used the standard multi-index notation. The summation is extended over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_N)$ whose order $|\alpha| = \sum_{i=1}^N \alpha_i$ does not exceed m).

Denote the set of functions satisfying conditions (a) and (b) by χ' .

(2) We define a topology in the set χ' as follows:

A sequence $\{f_n\}$ in χ' will be said to converge to a function $f(x)$ (written $f_n \xrightarrow{\chi} f$) if:

$$(a) \quad \|f_n - f\|_{2(\Omega')} \rightarrow 0$$

for each compact interior subregion Ω' , and

$$(b) \quad \|f_n - f\|_{1(\Omega_{\overset{\circ}{x}_N})} \rightarrow 0 \text{ for all } \overset{\circ}{x}_N \geq h.$$

Note that this implies $f \in \chi'$. We denote the resulting topological space by χ . For f in χ , $\bar{A}f$ is defined to be $-\Delta f$.

Let Λ denote all those real numbers contained in (v_1, ∞) other than the set $\{v_n\}$.

Theorem 2.1. Suppose $\lambda \in \Lambda$, $v_r < \lambda < v_{r+1}$, $\mu \in \chi$ and $\bar{A}\mu = \lambda\mu$. Suppose also that for each $\hat{x}_N \geq h$, $\mu \in L_2(l_{\hat{x}_N}^o)$, $\frac{\partial \mu}{\partial x_N} \in L_2(l_{\hat{x}_N}^o)$ and the following two conditions hold:

$$(2.1) \quad \int_{l_{\hat{x}_N}^o} \mu(\hat{x}_N, \tilde{x}) \eta_j(\tilde{x}) d\tilde{x} = c_j e^{-\sqrt{v_j - \lambda} \hat{x}_N}$$

$$(2.2) \quad \int_{l_{\hat{x}_N}^o} \frac{\partial \mu}{\partial x_N}(\hat{x}_N, \tilde{x}) \eta_j(\tilde{x}) d\tilde{x} = -c_j \sqrt{v_j - \lambda} e^{-\sqrt{v_j - \lambda} \hat{x}_N}$$

where for $j \leq r$, $\sqrt{v_j - \lambda}$ denotes $-i\sqrt{\lambda - v_j}$. Then $\mu \equiv 0$.

Note: Conditions (2.1) and (2.2) are called radiation conditions. They are the analogues of the Sommerfield radiation condition for the exterior problem.

Proof: We shall show that $\mu \in L_2(\Omega)$. This would imply that $\mu \equiv 0$ since A has no eigenvalues.

It follows from (2.1) and (2.2) that the Fourier expansion of $\mu(x)$ and $\frac{\partial \mu(x)}{\partial x_N}$ have the form:

$$(2.3) \quad \mu(x) = \sum_{j=1}^r c_j e^{i\sqrt{\lambda - v_j} x_N} \eta_j(\tilde{x}) + \sum_{j=r+1}^{\infty} c_j e^{-\sqrt{v_j - \lambda} x_N} \eta_j(\tilde{x})$$

$$\begin{aligned}
(2.4) \quad \frac{\partial \mu(x)}{\partial x_N} &= i \sum_{j=1}^r C_j \sqrt{\lambda - \nu_j} e^{i\sqrt{\lambda - \nu_j} x_N} \eta_j(\tilde{x}) \\
&- \sum_{j=r+1}^{\infty} C_j \sqrt{\nu_j - \lambda} e^{-\sqrt{\nu_j - \lambda} x_N} \eta_j(\tilde{x})
\end{aligned}$$

where $x_N \geq h$. Applying the divergence theorem to the functions $\mu(x)$ and $\bar{\mu}(x)$ and making use of the fact that $\mu = 0$ on $\bar{\Omega}$ and $\Delta \mu = -\lambda \mu$, we have

$$(2.5) \quad \int_{\partial_{x_N}} \left(\mu \frac{\partial \bar{\mu}}{\partial x_N} - \bar{\mu} \frac{\partial \mu}{\partial x_N} \right) d\tilde{x} = 0 \quad \text{for each } x_N \geq h.$$

Substituting the expressions (2.3) and (2.4) into (2.5), we obtain $2i \sum_{j=1}^r |C_j|^2 \sqrt{\lambda - \nu_j} = 0$. This implies $C_j = 0$ for $j = 1, \dots, r$. Hence,

$$\mu(\tilde{x}, x_N) = \sum_{j=r+1}^{\infty} C_j e^{-\sqrt{\nu_j - \lambda} x_N} \eta_j(\tilde{x}) \quad \text{for } x_N \geq h.$$

It thus follows that $\mu \in L_2(\Omega)$ and the theorem is proven. Q.E.D.

We shall next establish the existence of a solution to the boundary value problem $\bar{A}\mu = \lambda\mu + F$, where $\lambda \in \Lambda$. Set

$$(2.6) \quad \mu_{\varepsilon, \nu}(x) = (A - \lambda - \nu - i\varepsilon)^{-1} F(x) \quad \text{where } \varepsilon, \nu > 0.$$

In the following theorem we shall show that as $\varepsilon, \nu \rightarrow 0$, $\mu_{\varepsilon, \nu}$ converges in the sense of the space χ to the unique solution μ of this boundary value problem which satisfies conditions (2.1) and (2.2). The idea behind the proof is as follows: We first show that for each $\hat{x}_N \geq h$,

$$(2.7) \quad \|\mu_{\varepsilon, \nu}\|_{O(\Omega_{\hat{x}_N}^0)} \leq C$$

uniformly for all $\varepsilon > 0$, $\nu > 0$. We show this by assuming it is not true and deducing a contradiction. This is done by using elliptic estimates, Rellich's selection theorem, [3], and the Fourier expansions of the $\mu_{\varepsilon, \nu}$ to obtain a non-zero solution μ^* in χ of the equation $\bar{A}\mu = \lambda\mu^*$ which satisfies (2.1) and (2.2), thus contradicting Theorem 2.1. Having established inequality (2.7) we then repeat the above argument to obtain the required solution μ . We now give a precise statement and proof of Theorem 2.2.

Theorem 2.2. Suppose $\lambda \in \Lambda$ and the functions $\mu_{\varepsilon, \nu}(x)$ are defined by equation (2.6). Then there exists a function $\mu(x)$ such that $\mu_{\varepsilon, \nu} \xrightarrow{\chi} \mu$ as $\varepsilon, \nu \rightarrow 0$. μ satisfies the equation $\bar{A}\mu = \lambda\mu + F$. Also μ and $\frac{\partial \mu}{\partial x_N} \in L_2(\Omega_{\hat{x}_N}^0)$ and μ satisfies conditions (2.1) and (2.2) for each $\hat{x}_N \geq h$.

Proof: We first prove that (2.7) holds. (Throughout this paper we shall denote all constants by the same letter C).

Now assume that (2.7) does not hold for some $\dot{x}_N$. Then

$\exists$ a subsequence $\{\mu_{\varepsilon, \nu}\} \ni$:

$$\|\mu_{\varepsilon, \nu}\|_{\circ(\dot{x}_N)} \rightarrow \infty \text{ as } \varepsilon, \nu \rightarrow 0.$$

(In arguments of this type, we shall denote the members of the subsequence by the same letters as the members of the original sequence. Also $\|\mu\|_{j(\cap_{\dot{x}_N})}$ will be written as

$$\|\mu\|_{j(\dot{x}_N)}.) \text{ Set } \mu_{\varepsilon, \nu}^* = \frac{\mu_{\varepsilon, \nu}}{\|\mu_{\varepsilon, \nu}\|_{\circ(\dot{x}_N)}}.$$

$$(2.8) \quad \|\mu_{\varepsilon, \nu}^*\|_{\circ(\dot{x}_N)} = 1.$$

It follows from (2.6) that $\mu_{\varepsilon, \nu}^*$ satisfies the equation

$$(2.9) \quad A\mu_{\varepsilon, \nu}^* = (\lambda + \nu + i\varepsilon)\mu_{\varepsilon, \nu}^* + F_{\varepsilon, \nu}'$$

where $F_{\varepsilon, \nu}' = \frac{F}{\|\mu_{\varepsilon, \nu}\|_{\circ(\dot{x}_N)}}$. It is not hard to show ([4])

that for each $x_N < \dot{x}_N$:

$$(2.10) \quad \|\mu_{\varepsilon, \nu}^*\|_{1(x_N)} \leq C[\|\mu_{\varepsilon, \nu}^*\|_{\circ(\dot{x}_N)} + \|\Delta\mu_{\varepsilon, \nu}^*\|_{\circ(\dot{x}_N)}].$$

Using (2.8), (2.9) and (2.10) we have

$$(2.11) \quad \|\mu_{\varepsilon, \nu}^*\|_{1(x_N)} \leq C$$

uniformly for all ε, ν sufficiently small. By the Rellich selection theorem, $\exists$ a subsequence $\{\mu_{\varepsilon, \nu}^*\}$ which converges in the sense of $L_2(\Omega_{x_N})$ to some function μ^* , i.e.

$$(2.12) \quad \|\mu_{\varepsilon, \nu}^* - \mu^*\|_{0(x_N)} \rightarrow 0 \text{ as } \varepsilon, \nu \rightarrow 0.$$

Since $\|F_{\varepsilon, \nu}'\|_{0(\Omega)} \rightarrow 0$, we have from (2.9), (2.10) and (2.12):

$$(2.13) \quad \|\mu_{\varepsilon, \nu}^* - \mu^*\|_{1(x_N)} \rightarrow 0 \text{ for } x_N < \hat{x}_N.$$

We also have the following easily derived inequality:

$$(2.14) \quad \int_{\ell_{x_N}} |\mu_{\varepsilon, \nu}^*|^2 d\tilde{x} \leq \frac{2}{|x_N^1 - x_N^2|} \|\mu_{\varepsilon, \nu}^*\|_{0(x_N^1 x_N^2)}^2 \\ + 2|x_N^1 - x_N^2| \left\| \frac{\partial \mu_{\varepsilon, \nu}^*}{\partial x_N} \right\|_{0(x_N^1 x_N^2)}^2$$

where $x_N^1 \leq x_N \leq x_N^2$, $h \leq x_N$, and we denoted $\|\mu_{\varepsilon, \nu}^*\|_{0(\Omega_{x_N^1 x_N^2})}$

by $\|\mu_{\varepsilon, \nu}^*\|_{O_{(x_N^1 x_N^2)}}$. From (2.13) and (2.14) it follows

that $\mu_{\varepsilon, \nu}^*$ converges on each ℓ_{x_N} ($h \leq x_N \leq \dot{x}_N$) in the sense of $L_2(\ell_{x_N})$. By Lemma 2.1, we have:

$$(2.15) \quad \mu_{\varepsilon, \nu}^* = \sum_{j=1}^{\infty} C_j(\varepsilon, \nu) e^{-\sqrt{v_j - \lambda - \nu - i\varepsilon} x_N} \eta_j(\tilde{x})$$

for $x_N \geq h$ where $\operatorname{Re} \sqrt{v_j - \lambda - \nu - i\varepsilon} > 0$ for $j = 1, 2, \dots$.

The convergence on ℓ_h in the sense of $L_2(\ell_h)$ implies that for each j , $\exists C_j \ni : \sum_{j=1}^{\infty} |C_j(\varepsilon, \nu) - C_j|^2 \rightarrow 0$ as $\varepsilon, \nu \rightarrow 0$. We also have $\sqrt{v_j - \lambda - i\varepsilon - \nu} \rightarrow -i\sqrt{\lambda - v_j}$ for $j = 1, \dots, r$, and $\sqrt{v_j - \lambda - i\varepsilon - \nu} \rightarrow \sqrt{v_j - \lambda}$ for $j > r$ as $\varepsilon, \nu \rightarrow 0$. It follows from (2.15) that $\forall x_N \geq h$,

$$(2.16) \quad \|\mu_{\varepsilon, \nu}^* - \mu^*\|_{1(x_N)} \rightarrow 0$$

where

$$(2.17) \quad \mu^* = \sum_{j=1}^r C_j e^{i\sqrt{\lambda - v_j} x_N} \eta_j(\tilde{x}) + \sum_{j=r+1}^{\infty} C_j e^{-\sqrt{v_j - \lambda} x_N} \eta_j(\tilde{x}).$$

We now apply the following inequality, [11]:

$$\|\mu_{\varepsilon, \nu}^*\|_{2(\Omega'')} \leq C[\|\mu_{\varepsilon, \nu}^*\|_{O(\Omega')} + \|\Delta \mu_{\varepsilon, \nu}^*\|_{O(\Omega')}]$$

where $\Omega'' \subset \Omega'$ and Ω' is an arbitrary finite interior subregion of Ω . We thus have:

$$\mu_{\varepsilon, \nu}^* \xrightarrow{\chi} \mu^*.$$

Hence it follows that

$$(2.18) \quad \bar{A}\mu^* = \lambda\mu^*.$$

Since $\frac{\partial \mu_{\varepsilon, \nu}^*}{\partial x_N} = 0$ on $\overset{\circ}{\Omega}'_{x_N^1 x_N^2}$ whenever $h \leq x_N^1 \leq x_N^2$,

we may apply the preceding arguments to the functions $\frac{\partial \mu_{\varepsilon, \nu}^*}{\partial x_N}$ and $\frac{\partial \mu^*}{\partial x_N}$ to obtain the expansion.

$$(2.19) \quad \frac{\partial \mu_{\varepsilon, \nu}^*}{\partial x_N} = i \sum_{j=1}^r C_j \sqrt{\lambda - \nu_j} e^{i \sqrt{\lambda - \nu_j} x_N} \eta_j(\tilde{x}) \\ - \sum_{j=r+1}^{\infty} C_j \sqrt{\nu_j - \lambda} e^{-\sqrt{\nu_j - \lambda} x_N} \eta_j(\tilde{x})$$

where $x_N \geq h$.

It now follows from (2.17), (2.18), (2.19) and Theorem 2.1 that $\mu^* \equiv 0$. But this is impossible since $\|\mu_{\varepsilon, \nu}^*\|_{O(\overset{\circ}{x}_N)} = 1$

and $\|\mu_{\varepsilon, \nu}^* - \mu^*\|_{O(\overset{\circ}{x}_N)} \rightarrow 0$. This contradiction proves that

$\|\mu_{\varepsilon, \nu}\|_{O(\overset{\circ}{x}_N)} \leq C$ for each $\overset{\circ}{x}_N \geq h$.

It now follows, just as before, that $\exists$ a subsequence $\{\mu_{\varepsilon, \nu}\}$ and a function $\mu(x) \ni$:

$$\|\mu_{\varepsilon, \nu} - \mu\|_{1(x_N)} \rightarrow 0 \quad \forall \quad x_N \geq h.$$

Again $\mu_{\varepsilon, \nu} \xrightarrow{\chi} \mu$, μ and $\frac{\partial \mu}{\partial x_N} \in L_2(\ell_{x_N})$, and μ satisfies the conditions (2.1) and (2.2) as well as the equation $\bar{A}\mu = \lambda\mu + F$. Theorem 2.1 implies that the original sequence $\{\mu_{\varepsilon, \nu}\}$ converges in the sense of χ to μ . The theorem is thus proven. Q.E.D.

Note that in the proof of Theorem 2.2 we could just as easily have considered the functions $v_{\varepsilon, \nu} = (A - \lambda - \nu + i\varepsilon)^{-1}F$ and then taken the limit as $\varepsilon, \nu \rightarrow 0$ (again $\varepsilon > 0$). We would obtain in this way a different solution to the boundary value problem. We shall discuss this in greater detail in Chapter 3.

We next state a generalization of Theorem 2.2. We define Λ_1 to be the set of all complex numbers λ in the upper half plane ($\text{Im } \lambda \geq 0$) other than the set $\{v_n\}$. Denote by Λ_B an arbitrary bounded subset of Λ_1 . Instead of a single function $F(x)$, let us consider a sequence of functions $F_j(x)$ such that each $F_j \in C^\infty(\Omega)$ and $F_j(x) \equiv 0$ for $x_N \geq h'$ where $h' > h$. For each λ in Λ , let $\mu_j(x; \lambda)$ be the solution of the equation $\bar{A}\mu_j = \lambda\mu_j + F_j$ which satisfies the conditions (2.1) and (2.2). If λ is

non-real, set $\mu_j(x; \lambda) = (A - \lambda)^{-1} F_j$.

Theorem 2.3. (a) Suppose $\|F_j\|_{O(\Omega)} \leq C \quad \forall j = 1, 2, \dots$. Then $\|\mu_j(x; \lambda)\|_{O(\tilde{x}_N)} \leq C$ for each $\tilde{x}_N \geq h'$.

The constant C is independent of the function $\mu_j(x; \lambda)$ for each λ in Λ_B , $j = 1, 2, \dots$.

(b) Suppose $\|F_j - F_0\|_{O(\Omega)} \rightarrow 0$ and $\lambda_j \rightarrow \lambda_0$ as $j \rightarrow \infty$ where $\lambda_j \in \Lambda_B$ for $j = 0, 1, \dots$. Then $\|\mu_j(x; \lambda_j) - \mu_0(x; \lambda_0)\|_{O(\tilde{x}_N)} \rightarrow 0$ for each $\tilde{x}_N \geq h'$ as $j \rightarrow \infty$.

Theorem 2.3 can be proven in exactly the same way as Theorem 2.2. We are now in a position to construct the basic tools to be used in our expansion theorem. These are the generalized eigenfunctions of the operator A , to be denoted by $w_n(x; \xi)$. $w_n(x; \xi)$ will be constructed as the sum of a plane wave $\sin \xi x_N \eta_n(\tilde{x})$ and an outgoing wave $v_n(x; \xi)$, where the outgoing wave will satisfy the radiation conditions (2.1) and (2.2) at infinity. We shall construct these $v_n(x; \xi)$ as solutions of the boundary value problem: $(-\Delta - \xi^2 - v_n) v_n(x; \xi) = 0$, $v_n(x; \xi) = -\sin \xi x_N \eta_n(\tilde{x})$ on $\tilde{\Omega}$. To do this we use Theorem 2.2 and a standard argument for converting a homogeneous equation with non-zero boundary conditions into an inhomogeneous equation with zero boundary conditions.

Let $\zeta(x_N)$ be a "cutoff" function satisfying the following conditions:

$$(a) \quad \zeta(x_N) \in C^\infty(0, \infty)$$

$$(b) \quad \zeta(x_N) \equiv 0 \quad \text{for } x_N \geq h'$$

$$(c) \quad \zeta(x_N) \equiv 1 \quad \text{for } h \geq x_N.$$

Set $G_n(x; \xi) = \zeta(x_N) \sin \xi x_N \eta_n(\tilde{x})$ and $F_n(x; \xi) = (-\Delta - \xi^2 - v_n)G_n(x; \xi)$. Suppose $\xi^2 + v_n \in \Lambda$. By Theorems 2.1 and 2.2, $\exists$ a unique solution $\mu_n(x; \xi) \in \chi$ of the equation $\bar{A}\mu_n(x; \xi) = (\xi^2 + v_n)\mu_n(x; \xi) + F_n(x; \xi)$ such that $\mu_n(x; \xi)$ satisfies (2.1) and (2.2). Setting $v_n(x; \xi) = \mu_n(x; \xi) - G_n(x; \xi)$, we see that $v_n(x; \xi)$ is the required outgoing solution. Set

$$(2.20) \quad w_n(x; \xi) = \sin \xi x_N \eta_n(\tilde{x}) + v_n(x; \xi).$$

$w_n(x; \xi) = 0$ on $\overset{\circ}{\Omega}$ and $w_n(x; \xi)$ satisfies the equation $(\Delta + \xi^2 + v_n)w_n(x; \xi) = 0$. In Section 3 it will be shown that the functions $w_n(x; \xi)$ form a complete set of generalized eigenfunctions for the operator A . Note that $w_n(x; \xi)$ is not identically zero since $v_n(x; \xi)$ satisfies (2.1).

We shall also need the following set of functions. Suppose κ is a complex number such that $\text{Im } \kappa > 0$. Set

$$(2.21) \quad \mu_n(x; \xi; \kappa) = (A - \kappa)^{-1} F_n(x; \xi; \kappa)$$

where

$$(2.22) \quad F_n(x; \xi; \kappa) = (-\Delta - \kappa) G_n(x; \xi)$$

and

$$(2.23) \quad G_n(x; \xi) = \zeta(x_N) \sin \xi x_N \eta_n(\tilde{x}).$$

Set

$$(2.24) \quad v_n(x; \xi; \kappa) = \mu_n(x; \xi; \kappa) - G_n(x; \xi)$$

and

$$(2.25) \quad w_n(x; \xi; \kappa) = v_n(x; \xi; \kappa) + \sin \xi x_N \eta_n(\tilde{x}).$$

Note: (1) It follows from Lemma 2.1 that $v_n(x; \xi; \kappa)$ dies down exponentially at infinity.

(2) $w_n(x; \xi; \kappa) = 0$ on $\tilde{\Omega}$ and satisfies the equation $(-\Delta - \xi^2 - v_n)w_n(x; \xi; \kappa) = (\kappa - \xi^2 - v_n)v_n(x; \xi; \kappa)$.

(3) $w_n(x; \xi; \kappa)$ is an "approximate eigenfunction" for close to $\xi^2 + v_n$ in the sense that $w_n(x; \xi; \kappa) \xrightarrow{\chi} w_n(x; \xi)$ as $\kappa \rightarrow \xi^2 + v_n$. This follows from Theorem 2.2.

In the next theorem we prove that the functions $v_n(x; \xi; \kappa)$ and their derivatives are bounded uniformly by powers of ξ and v_n .

Theorem 2.4. Consider the set of complex numbers κ in Λ_B such that for each positive integer j , we have $\operatorname{Re} \sqrt{v_j - \kappa} \neq 0$. We choose that square root for which $\operatorname{Re} \sqrt{v_j - \kappa} > 0 \quad \forall j = 1, 2, \dots$. Then

(a) For each multiple index α , $\exists$ an integer M_α and a constant C_α such that

$$(2.26) \quad \max_{\Omega_{x_N}^0} |D^\alpha v_n(x; \xi; \kappa)| \leq C_\alpha (\xi^2 + v_n)^{M_\alpha}$$

where C_α depends on x_N but is independent of ξ in $[0, \infty)$, $n = 1, 2, \dots$, and κ in Λ_B . Here D^α signifies differentiation with respect to x .

(b) For each complex number κ such that $\operatorname{Im} \kappa > 0$ and each multiple index α , $\exists$ constants C_α and $d_\alpha > 0$ and an integer M_α such that for all points x in Ω for κ which x_N is sufficiently large we have

$$(2.27) \quad |D^\alpha v_n(x; \xi; \kappa)| \leq (\xi^2 + v_n)^{M_\alpha} C_\alpha e^{-d_\alpha x_N}$$

where again the constants do not depend on ξ and n for ξ in $[0, \infty)$, and $n = 1, 2, \dots$. They will, however, depend on κ .

Proof: It follows from the theory of elliptic equations ([1]) that for each w in $\mathring{H}_1(B)$, where B is a bounded domain with sufficiently smooth boundary, we have

$$\|w\|_{M+2(B)} \leq C[\|\Delta w\|_{M(B)} + \|w\|_{0(B)}] \quad \text{provided} \quad \|\Delta w\|_{M(B)} < \infty,$$

where M is any non-negative integer. We apply this result to the functions $\Gamma(x_N)\mu(x;\xi;\kappa)$ where $\Gamma(x_N)$ is a "cutoff" function of the type used before, i.e.

$\Gamma(x_N) \equiv 1$ for $x_N \leq \hat{x}_N$, $\Gamma(x_N) \equiv 0$ for $\hat{x}_N + \varepsilon \leq x_N$ ($\varepsilon > 0$), and $\Gamma(x_N) \in C^\infty(0, \infty)$. Using Theorem 2.3, and Definitions (2.21), (2.22), and (2.23), it follows that

$$\|\mu_n(x;\xi;\kappa)\|_{0(\hat{x}_N+\varepsilon)} \leq C(\xi^2 + v_n)$$

and

$$\|F_n(x;\xi;\kappa)\|_{M(\hat{x}_N+\varepsilon)} \leq C(\xi^2 + v_n)^{M+2}$$

for each $M = 0, 1, \dots$. It now follows by an induction argument that for each M , $\exists$ an integer S_M , $\ni$

$$(2.28) \quad \|\mu_n(x;\xi;\kappa)\|_{M(\hat{x}_N)} \leq C(\xi^2 + v_n)^{S_M}.$$

Applying Sobolev's theorem [16], and (2.28) we obtain (2.26).

Thus, part (a) is proven.

(b) First we assert that the following estimate holds:

$$(2.29) \quad \max_{\tilde{x} \in \ell} |D^\alpha \eta_j(\tilde{x})| \leq C_\alpha v_j^{|\alpha|+N}.$$

This follows from an argument similar to that used in part (a) since in the bounded $N-1$ dimensional domain ℓ , the normalized eigenfunctions $\eta_j(\tilde{x})$ satisfy the elliptic boundary value problem:

$$\sum_{i=1}^{N-1} \frac{\partial^2 \eta_j(\tilde{x})}{\partial x_i^2} = -v_j \eta_j(\tilde{x}),$$

$$\eta_j(\tilde{x}) = 0 \quad \text{on } \ell.$$

Next we note that by Lemma 2.1 we have:

$$(2.30) \quad v_n(x; \xi; K) = \sum_{j=1}^{\infty} C_j(\xi, n) e^{-\sqrt{v_j - K} x_N} \eta_j(\tilde{x})$$

where $x_N \geq h'$.

We estimate the $C_j(\xi_n)$ in terms of $\xi^2 + v_n$. Using the expression (2.30) for $v_n(x; \xi; K)$, we square its modulus and integrate over that part of Ω in which $x_N \geq h'$. We thus have:

$$(2.31) \quad \int_{\ell} \int_{h'}^{\infty} |v_n(x; \xi; K)|^2 dx_N d\tilde{x} = \frac{\pi}{2} \sum_{j=1}^{\infty} \frac{|C_j(\xi, n)|^2 e^{-2 \operatorname{Re} \sqrt{v_j - K} h'}}{2 \operatorname{Re} \sqrt{v_j - K}}.$$

From (2.24) we have:

$$\begin{aligned}
 (2.32) \quad \int_{\ell} \int_{h'}^{\infty} |v_n(x; \xi; \kappa)|^2 dx_N d\tilde{x} &\leq \|v_n\|_{o(\Omega)}^2 \leq \\
 &\leq C[\|G_n\|_{o(\Omega)}^2 + \|\mu_n\|_{o(\Omega)}^2].
 \end{aligned}$$

From the definitions (2.21), (2.22), and (2.23), it follows that:

$$\begin{aligned}
 (2.33) \quad \|G_n(x; \xi)\|_{o(\Omega)}^2 + \|\mu_n(x; \xi; \kappa)\|_{o(\Omega)}^2 \\
 \leq C + |||(A - \kappa)^{-1}||| \|F_n(x; \xi; \kappa)\|_{o(\Omega)}^2 \\
 \leq C(\xi^2 + v_n)
 \end{aligned}$$

where $|||(A - \kappa)^{-1}|||$ denotes the operator norm of $(A - \kappa)^{-1}$. Thus we conclude from (2.31), (2.32), and (2.33) that:

$$(2.34) \quad |C_j(\xi, n)| \leq C(\xi^2 + v_n) \operatorname{Re} \sqrt{v_j - \kappa} e^{\operatorname{Re} \sqrt{v_j - \kappa} h'}.$$

Finally (2.29), (2.30), and (2.34) imply estimate (2.27). This completes the proof of the theorem. Q.E.D.

CHAPTER 3 PROOF OF THE EXPANSION THEOREM

In Chapter 1 we constructed a unitary transformation T_0 from $L_2(S)$ onto the space H described in Definition 1.3. Recall that for each function $\psi(x)$ in $C_0^\infty(S)$, $T_0\psi$ was defined in terms of the generalized eigenfunctions $w_n^0(x; \xi) = \sin \xi x_N \eta_n(\tilde{x})$ of the operator A_0 as follows: $T_0\psi = \{\hat{\psi}_n^0(\xi)\}$ where:

$$(3.1) \quad \hat{\psi}_n^0(\xi) = (\psi, \tilde{w}_n^0(\cdot; \xi))_S = \int_S \psi(x) \sin \xi x_N \overline{\eta_n(\tilde{x})} dx.$$

In this chapter we shall prove an analogous result for the domain Ω . That is, a unitary transformation T from $L_2(\Omega)$ onto H will be constructed using the eigenfunctions $w_n(x; \xi)$ defined by (2.20). It will be shown that T diagonalizes the operator A . The main results of this paper are embodied in Theorems 3.5 and 3.6.

We begin by defining the Fourier transform of a function $\phi(x)$ in $C_0^\infty(\Omega)$ and proving that it has two properties.

Definition 3.1. Suppose $\phi \in C_0^\infty(\Omega)$. Set

$$(3.2) \quad \hat{\phi}_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2} (\phi, w_n(\cdot; \xi))_\Omega = \left(\frac{2}{\pi}\right)^{1/2} \int_\Omega \phi(x) \overline{w_n(x; \xi)} dx.$$

For each complex number K such that $\text{Im } K > 0$, set

$$(3.3) \quad \hat{\phi}_n(\xi; \kappa) = \left(\frac{2}{\pi}\right)^{1/2} (\phi, w_n(\cdot; \xi; \kappa))_{\Omega} =$$

$$\left(\frac{2}{\pi}\right)^{1/2} \int_{\Omega} \phi(x) \overline{w_n(x; \xi; \kappa)} \, dx.$$

The $\hat{\phi}_n(\xi; \kappa)$ are "approximate transforms" in the sense that

$$(3.4) \quad \hat{\phi}_n(\xi; \kappa) \rightarrow \hat{\phi}_n(\xi) \quad \text{as} \quad \kappa \rightarrow \xi^2 + v_n.$$

This will follow from the next theorem.

Theorem 3.1. Suppose $\phi(x) \in C_0^{\infty}(\Omega)$. Then (a) $\hat{\phi}_n(\xi; \kappa)$ is continuous as a function of κ for fixed ξ and n , where $\kappa \in \Lambda_1$. In particular, (3.4) holds for such κ . (b) $|\hat{\phi}_n(\xi; \kappa)| \leq (\xi^2 + v_n)C$ where C is a constant independent of ξ , n , and κ for ξ in $[0, \infty)$, $n = 1, 2, \dots$, and all κ in Λ_B .

Proof: (a) Suppose $|\kappa_j - \kappa_0| \rightarrow 0$ where $\kappa_j \in \Lambda_1$, $j = 0, 1, \dots$. Set $B = \text{support of } \phi(x)$.

$$\begin{aligned} (3.5) \quad & |\hat{\phi}_n(\xi; \kappa_j) - \hat{\phi}_n(\xi; \kappa_0)| \\ &= \left| \int_{\Omega} \phi(x) [\overline{w_n(x; \xi; \kappa_j)} - \overline{w_n(x; \xi; \kappa_0)}] \, dx \right| \\ &\leq \int_B |\phi(x) [\overline{w_n(x; \xi; \kappa_j)} - \overline{w_n(x; \xi; \kappa_0)}]| \, dx \end{aligned}$$

$$\begin{aligned}
&\leq \| \phi \|_{L_2(B)} \| w_n(x; \xi; \kappa_j) - w_n(x; \xi; \kappa_0) \|_{L_2(B)} \\
&= C \| w_n(x; \xi; \kappa_j) - w_n(x; \xi; \kappa_0) \|_{L_2(B)}.
\end{aligned}$$

Now from the Definition (2.25) of $w_n(x; \xi; \kappa)$, Theorem 2.3(b), and Estimate (3.5), it follows that:

$$\begin{aligned}
| \hat{\phi}_n(\xi; \kappa_j) - \hat{\phi}_n(\xi; \kappa_0) | &\leq C \| w_n(x; \xi; \kappa_j) - w_n(x; \xi; \kappa_0) \|_{L_2(B)} \\
&= C \| \mu_n(x; \xi; \kappa_j) - \mu_n(x; \xi; \kappa_0) \|_{L_2(B)} \rightarrow 0.
\end{aligned}$$

Setting $\kappa_0 = \xi^2 + v_n$, we see that (3.4) holds. This proves part (a).

(b) This part can be proven in the same way as part (a). In this case we would make use of Theorem 3.2 (a). Q.E.D.

Now suppose that $\psi(x) \in C_0^\infty(S)$. Denote by ψ_Ω the restriction of $\psi(x)$ to the domain Ω . We shall construct $R_\kappa(\psi_\Omega) = (A - \kappa)^{-1} \psi_\Omega$ where κ is a complex number with positive imaginary part. Set

$$(3.6) \quad R_\psi(x; \kappa) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\hat{\psi}_n^0(\xi) w_n(x; \xi; \kappa) d\xi}{\xi^2 + v_n - \kappa}$$

for each x in Ω .

Theorem 3.2. For each $\psi(x)$ in $C_0^\infty(S)$:

$$(3.7) \quad R_{\psi}(x; \kappa) = R_{\kappa}(\psi_{\Omega}) .$$

Proof: From the Definition (2.25) of $w_n(x; \xi; \kappa)$, it follows that $R_{\psi}(x; \kappa) = R_{\psi}^{(1)}(x; \kappa) + R_{\psi}^{(2)}(x; \kappa)$ where

$$(3.8) \quad R_{\psi}^{(1)}(x; \kappa) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\hat{\psi}_n^0(\xi) \sin \xi x_N \eta_n(\tilde{x})}{\xi^2 + \nu_n - \kappa} d\xi$$

and

$$(3.9) \quad R_{\psi}^{(2)}(x; \kappa) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\hat{\psi}_n^0(\xi) \nu_n(x; \xi; \kappa)}{\xi^2 + \nu_n - \kappa} d\xi .$$

Integration by parts on the right hand side of equation (3.1) yields

$$(3.10) \quad |D^{\alpha} \hat{\psi}_n^0(\xi)| = O((\xi^2 + \nu_n)^{-\ell})$$

for each $\ell = 0, 1, \dots$, and $\alpha = 0, 1, \dots$ where D^{α} denotes differentiation with respect to ξ . It also follows, using asymptotic expressions for the eigenvalues of the operator A_{ℓ} [2] that there exists an integer M such that

$$(3.11) \quad \sum_{j=1}^{\infty} \frac{1}{\nu_j^M} < \infty .$$

Now consider the series (3.8) defining $R_{\psi}^{(1)}(x; \kappa)$. Using (3.10) and (3.11) it is easy to see that we may differentiate

this series termwise any number of times. Also, an integration by parts in (3.8) gives the estimate:

$$(3.12) \quad |D^\alpha R_\psi^{(1)}(x; \kappa)| = O(x_N^{-2})$$

where $x \in \Omega$ and D^α represents any derivative with respect to the variables $x_1, \dots, x_N$. Next consider $R_\psi^{(2)}(x; \kappa)$. Applying (3.10), (3.11), and Theorem 2.4, we see that the series (3.9) may be differentiated termwise any number of times and that the following estimate holds:

$$(3.13) \quad |D^\alpha R_\psi^{(2)}(x; \kappa)| \leq C_\alpha e^{-d_\alpha x_N} \quad (d_\alpha > 0)$$

for all x in Ω and x_N large enough. From (3.12) and (3.13) it follows that $D^\alpha R_\psi(x; \kappa) \in L_2(\Omega)$ for every derivative D^α .

Now apply the operator $-\Delta - \kappa$ to the series (3.6).

$$(-\Delta - \kappa)R_\psi(x; \kappa) = \sum_{n=1}^{\infty} \int_0^\infty \frac{\hat{\psi}_n^0(\xi)(-\Delta - \kappa)w_n(x; \xi; \kappa)}{\xi^2 + v_n - \kappa} d\xi.$$

We conclude from Theorem 1.1 and the equation $(-\Delta - \kappa)w_n(x; \xi; \kappa) = (\xi^2 + v_n - \kappa)w_n^0(x; \xi)$ that $(-\Delta - \kappa)R_\psi(x; \kappa) = \psi(x) \quad \forall x$ in Ω . Also, $R_\psi(x; \kappa) = 0$ on $\bar{\Omega}$ since $w_n(x; \xi; \kappa) = 0$ on $\bar{\Omega}$. Hence equation (3.7) follows. Q.E.D.

Next, let $\{E_\lambda\}$ represent the spectral resolution associated with the self-adjoint operator A . It follows from the spectral theorem that $E_{\lambda+} = E_\lambda$ and $A = \int \lambda dE_\lambda$.

We state without proof the well-known formula ([17]):

$$(3.14) \quad \frac{1}{2}((E_b + E_{b-})f, f)_{(\Omega)} - \frac{1}{2}((E_a + E_{a-})f, f)_{(\Omega)} \\ = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \|R_{\tau - i\varepsilon}\|_{L_2(\Omega)}^2 d\tau.$$

Equation (3.14) holds for each f in $L_2(\Omega)$. We shall use (3.14) to prove the following important result:

$$(3.15) \quad \frac{1}{2}((E_b + E_{b-})\phi, \phi)_{(\Omega)} - \frac{1}{2}((E_a + E_{a-})\phi, \phi)_{(\Omega)} \\ = \sum_{a < \xi^2 + \nu_n < b} \int |\hat{\phi}_n(\xi)|^2 d\xi \quad \forall \phi \text{ in } C_0^\infty(\Omega),$$

where $[a, b] \in \Lambda$. Once this result is established, the expansion theorem will follow immediately.

Making use of (3.14), the proof of (3.15) reduces to the evaluation of

$$\lim_{\varepsilon \downarrow 0} \int_a^b \|R_{\tau - i\varepsilon} \phi\|_{L_2(\Omega)}^2 d\tau.$$

We begin by employing Theorem 3.2 to obtain an expression for the integrand, $\|R_{\tau - i\varepsilon} \phi\|_{L_2(\Omega)}^2$.

Theorem 3.3. For each ϕ in $C_0^\infty(\Omega)$, we have

$$(3.16) \quad \|R_{\overline{\kappa}}\phi\|_{L_2(\Omega)}^2 = \sum_{n=1}^{\infty} \int_0^{\infty} |(\xi^2 + v_n - \overline{\kappa})^{-1} \hat{\phi}_n(\xi; \kappa)|^2 d\xi$$

where $\text{Im } \kappa > 0$.

Proof: For each ψ in $C_0^\infty(S)$, it follows from Theorem 3.2 and the fact that R_κ and $R_{\overline{\kappa}}$ are adjoints that:

$$(3.17) \quad (R_{\overline{\kappa}}\phi, \psi)_\Omega = (\phi, R_\kappa(\psi))_\Omega = (\phi, R_\psi(\cdot; \kappa))_\Omega \\ = \int_\Omega \phi(x) \overline{R_\psi(x; \kappa)} dx .$$

Using the definition, (3.6), of $R_\psi(x; \kappa)$, we get:

$$(3.18) \quad \int_\Omega \phi(x) \overline{R_\psi(x; \kappa)} dx = \\ \int_\Omega \phi(x) \left(\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\overline{\hat{\psi}_n^0(\xi)} \overline{w_n(x; \xi; \kappa)}}{\xi^2 + v_n - \overline{\kappa}} d\xi \right) dx .$$

We may now interchange the order of integration and summation in (3.18) using Fubini's theorem, the definition (2.25) of $w_n(x; \xi; \kappa)$, and the properties of $v_n(x; \xi; \kappa)$ and $\hat{\psi}_n^0(\xi)$ given by Theorem 2.4 and Estimate (3.10), respectively.

Hence:

$$(3.19) \quad \int_{\Omega} \phi(x) \overline{R_{\psi}(x; \kappa)} dx = \sum_{n=1}^{\infty} \int_0^{\infty} (\xi^2 + v_n - \overline{\kappa})^{-1} \hat{\psi}_n^0(\xi) \hat{\phi}_n(\xi; \kappa) d\xi$$

where we also used the definition, (3.3), of $\hat{\phi}(\xi; \kappa)$. For each x in $S - \Omega$, set $R_{\overline{\kappa}} \phi(x) = 0$. Combining (3.17) and (3.19), we have

$$(3.20) \quad (R_{\overline{\kappa}} \phi, \psi)_S = \sum_{n=1}^{\infty} \int_0^{\infty} (\xi^2 + v_n - \overline{\kappa})^{-1} \hat{\phi}_n(\xi; \kappa) \hat{\psi}_n^0(\xi) d\xi$$

$$= ((\xi^2 + v_n - \overline{\kappa})^{-1} \hat{\phi}_n(\xi; \kappa), \hat{\psi}_n^0(\xi))_H.$$

The unperturbed expansion theorem gives

$$(3.21) \quad (R_{\overline{\kappa}} \phi, \psi)_S = ((R_{\overline{\kappa}} \phi)^{\hat{0}}, \hat{\psi}^0)_H.$$

Since the set of functions $\psi(x)$ chosen are dense in $L_2(S)$, the set of elements $\hat{\psi}^0$ will be dense in H . Comparing (3.20) and (3.21), we conclude

$$(3.22) \quad (R_{\overline{\kappa}} \phi)^{\hat{0}}(\xi) = \{(\xi^2 + v_n - \overline{\kappa})^{-1} \hat{\phi}_n(\xi; \kappa)\}.$$

Since $R_{\overline{\kappa}} \phi(x) = 0$ in $S - \Omega$, equation (3.22) and Theorem 1.1 imply

$$\|R_{\overline{\kappa}} \phi\|_{L_2(\Omega)}^2 = \|R_{\overline{\kappa}} \phi\|_{L_2(S)}^2 = \|(R_{\overline{\kappa}} \phi)^{\hat{0}}\|_H^2$$

$$= \|(\xi^2 + v_n - \overline{\kappa})^{-1} \hat{\phi}_n(\xi; \kappa)\|_H^2 = \sum_{n=1}^{\infty} \int_0^{\infty} |(\xi^2 + v_n - \overline{\kappa})^{-1} \hat{\phi}_n(\xi; \kappa)|^2 d\xi.$$

This completes the proof of the theorem. Q.E.D.

Before completing the proof of (3.15), it will be necessary to establish three lemmas. The purpose of these lemmas is to show that for each $\phi(x)$ in $C_0^\infty(\Omega)$

$$(3.23) \quad \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{\xi^2 + v_n \geq b+1} \int_a^b \int \frac{|\hat{\phi}_n(\xi; \kappa)|^2 d\tau d\xi}{(\xi^2 + v_n - \tau)^2 + \varepsilon^2} = 0$$

where $\kappa = \tau + i\varepsilon$. Equation (3.23) corresponds to a result proven by Shenk [14] for the exterior problem.

Let $\mathcal{S}$ be the set of functions $\psi(x)$ defined in S and satisfying the condition that $A_0^m \psi \in D(A_0)$, i.e. $\Delta^m \psi \in L_2(S)$ and $\Delta^m \psi = 0$ on $\overset{\circ}{S}$ for each $m = 0, 1, \dots$.

Lemma 3.1. The set of elements of the form $\{(1 + \xi^2 + v_n) \hat{\psi}_n^0(\xi)\}$, where ψ is an arbitrary function in $\mathcal{S}$, is dense in H .

Proof: As a consequence of Theorem 1.1, it is sufficient to show that the set of functions $(A_0 + I)\mathcal{S}$ is dense in $L_2(S)$. We shall show that $C_0^\infty(S) \subseteq (A_0 + I)\mathcal{S}$. Suppose $\phi(x)$ is any function in $C_0^\infty(S)$. Set

$$(3.24) \quad \rho(x) = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\hat{\phi}_n^0(\xi) \sin \xi x_N \eta_n(\tilde{x})}{\xi^2 + v_n + 1} d\xi.$$

It follows from (3.10) and (3.11) that the series (3.24) may be differentiated termwise any number of times. From this and Theorem 1.1 it is obvious that $\rho \in \mathcal{S}$ and

$(A_0 + I)\rho = \sigma$. This shows that $(A_0 + I)\mathcal{Y}$ contains a dense subset of $L_2(S)$. Hence the lemma is proven. Q.E.D.

Before stating the next lemma, we make some definitions. Define $\Lambda_{[a,b]}$ to be the set of complex numbers $\kappa = \tau + i\varepsilon$ satisfying the conditions

$$(1) \quad 0 < a \leq \tau \leq b < \infty, \quad \text{where } [a,b] \in \Lambda$$

$$(2) \quad 0 < \varepsilon \leq \varepsilon_0.$$

We denote by $G_\kappa(x;y)$, Green's function for the operator $A - \kappa$. Unless otherwise stipulated, it is to be understood that $\kappa \in \Lambda_{[a,b]}$.

Lemma 3.2. Suppose B is a compact interior subset of Ω . Then

$$(3.25) \quad \max_{\substack{x \in \Omega'_h \\ y \in B}} \left| \frac{\partial G_\kappa(x;y)}{\partial n} \right| \leq C,$$

where $\frac{\partial}{\partial n}$ represents the normal derivative on Ω . It is important that the constant C be independent of κ for all κ in $\Lambda_{[a,b]}$.

Proof: Let $G_\kappa^0(x;y)$ denote Green's function for the operator $A_Z - \kappa$ where Z is the infinite cylinder $-\infty < x_N < \infty$, $\tilde{x} \in \ell$. It follows from the results in a paper by Tikhonov and Samarskiy, [13], that

$$(3.26) \quad G_\kappa^0(x;y) = \sum_{n=1}^{\infty} \frac{\eta_n(\tilde{x}) \eta_n(\tilde{y}) e^{-\sqrt{v_n - \kappa} |x_N - y_N|}}{\sqrt{v_n - \kappa}}$$

and

$$(3.27) \quad \max_{\substack{x \in \overset{\circ}{\Omega}_h \\ y \in B}} \left| \frac{\partial G_K^0(x; y)}{\partial n} \right| \leq C$$

where C is independent of K for all K in $\Lambda_{[a, b]}$. We shall prove estimate (3.25) by showing that $G_K(x; y) = G_K^0(x; y) + H_K(x; y)$ where $H_K(x; y)$ satisfies the equation

$$(3.28) \quad (-\Delta_x - K)H_K = 0$$

(Δ_x denotes differentiation with respect to the variable point x), the boundary condition

$$(3.29) \quad H_K(x; y) = -G_K^0(x; y) \text{ on } \overset{\circ}{\Omega},$$

and the inequality

$$(3.30) \quad \max_{\substack{x \in \overset{\circ}{\Omega}_h \\ y \in B}} \left| \frac{\partial H_K(x; y)}{\partial n} \right| \leq C.$$

We construct $H_K(x; y)$ by means of the unperturbed expansion theorem and the functions $v_n(x; \xi; K)$.

Suppose $\psi(x) \in \mathcal{G}$. Set

$$(3.31) \quad U_\psi(x; K) = \sum_{n=1}^{\infty} \int_0^{\infty} \hat{\psi}_n^0(\xi) v_n(x; \xi; K) d\xi.$$

We may differentiate this series termwise. Since $v_n(x; \xi; \kappa)$ satisfies the equation $(-\Delta - \kappa)v_n(x; \xi; \kappa) = 0$, we have

$$(3.32) \quad (-\Delta - \kappa)U_\psi(x; \kappa) = 0.$$

From the boundary condition $v_n(x; \xi; \kappa) = -w_n^0(x; \xi)$ on $\overset{\circ}{\Omega}$, and Theorem 1.1, it follows that

$$(3.33) \quad U_\psi(x;) = -\psi(x) \text{ on } \overset{\circ}{\Omega}.$$

We now construct the functions $H_\kappa(x; y)$ as follows: Let $\Gamma(x)$ be a function satisfying the following conditions:

- (a) $\Gamma(x) = 0$ in a neighborhood of B
- (b) $\Gamma(x) = 1$ in a neighborhood of $\overset{\circ}{\Omega}'_h$
- (c) $\Gamma(x) = 0$ in a neighborhood of $x_N = 0$

and for $x_N > h'$

- (d) for each $x_N < \infty$ and $\tilde{x} \in \overset{\circ}{\Omega}$, we have $D^\alpha \Gamma(x_N, \tilde{x}) = 0$ where D^α represents any derivative with respect to the variables $x_1, \dots, x_{N-1}$.

It is easily seen that $\Gamma(x)G_\kappa^0(x; y) \in \mathcal{S}$ for each fixed point y in B . Thus we may construct the functions $U\Gamma G_\kappa^0(x; \kappa)$. From equations (3.32) and (3.33), it is clear that $U\Gamma G_\kappa^0(x; \kappa)$ satisfies the equation $(-\Delta - \kappa)U\Gamma G_\kappa^0 = 0$, and the boundary condition $U\Gamma G_\kappa^0(x; \kappa) = -G_\kappa^0(x; y)$ on $\overset{\circ}{\Omega}$.

It now follows that

$$U \Gamma G_K^0(x; \kappa) = H_K(x; y)$$

that is

$$G_K(x; y) = G_K^0(x; y) + U \Gamma G_K^0(x; y) .$$

We shall estimate the first order derivatives $D_x U \Gamma G_K^0(x; \kappa)$ where $y \in B$. We assert that for each $m = 0, 1, \dots$

$$(3.34) \quad \sum_{n=1}^{\infty} \int_0^{\infty} |(\xi^2 + v_n)^m (\Gamma G_K^0)_n^{\wedge}(\xi)|^2 d\xi \leq C$$

where C is independent of κ in $\Lambda_{[a,b]}$ as well as the points y in B . This follows from the unperturbed expansion theorem and the fact that $\|\Delta_x^m(\Gamma(x)G_K^0(x; y))\|_{L_2(S)}^2 \leq C$. Using (3.34), the Definition (3.31) of $U_{\psi}(x; \kappa)$, the properties of $v_n(x; \xi; \kappa)$ given by Theorem 3.4 (a), and the Schwarz inequality, we have

$$(3.35) \quad \max_{\substack{x \in \bar{\Omega}_h \\ y \in B}} |DU \Gamma G_K^0(x; \kappa)| = \max_{\substack{x \in \bar{\Omega}_h \\ y \in B}} |DH_K(x; y)| \leq C,$$

C independent of κ in $\Lambda_{[a,b]}$ as well as points y in B . Combining (3.27) and (3.35), we conclude that (3.25) holds. Q.E.D.

We now establish equation (3.23).

Lemma 3.3. For each ϕ in $C_0^\infty(\Omega)$,

$$(3.23) \quad \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{\xi^2 + v_n > b+1} \int_a^b \int \frac{|\hat{\phi}_n(\xi; \kappa)|^2 d\tau d\xi}{(\xi^2 + v_n - \tau)^2 + \varepsilon^2} = 0$$

where $\kappa = \tau + i\varepsilon$ and $\kappa \in \Lambda_{[a,b]}$.

Proof: Using the Definitions (3.1) and (3.3), of $\hat{\phi}_n^0(\xi)$ and $\hat{\phi}_n(\xi; \kappa)$, respectively, it follows that

$$(3.36) \quad \hat{\phi}_n(\xi; \kappa) - \hat{\phi}_n^0(\xi) = \int_{\Omega} \overline{v_n(x; \xi; \kappa)} \phi(x) dx.$$

Let $\psi(x)$ be an arbitrary function in $\mathcal{G}$. Multiply both sides of Equation (3.36) by $\hat{\psi}_n^0(\xi)$, then integrate with respect to ξ and sum with respect to n .

$$\begin{aligned} (3.37) \quad & \sum_{n=1}^{\infty} \int_0^{\infty} \hat{\psi}_n^0(\xi) [\overline{\hat{\phi}_n(\xi; \kappa) - \hat{\phi}_n^0(\xi)}] d\xi \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} \hat{\psi}_n^0(\xi) \left[\int_{\Omega} v_n(x; \xi; \kappa) \overline{\phi(x)} dx \right] d\xi \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} \hat{\psi}_n^0(\xi) \left[\int_B v_n(x; \xi; \kappa) \overline{\phi(x)} dx \right] d\xi \end{aligned}$$

where $B = \text{support of } \phi(x)$. Using Fubini's theorem, Estimate (3.10), and Theorem 2.4 (a), we may interchange the

order of integration and summation on the right hand side.
Thus, (3.31) and (3.37) imply

$$(3.38) \quad \sum_{n=1}^{\infty} \int_0^{\infty} \hat{\psi}_n^0(\xi) [\overline{\hat{\phi}_n^0(\xi) - \hat{\phi}_n^0(\xi)}] d\xi = \int_B \overline{\phi(x)} U_{\psi}(x; \kappa) dx .$$

Hence

$$(3.39) \quad \left| \sum_{n=1}^{\infty} \int_0^{\infty} \hat{\psi}_n^0(\xi) [\overline{\hat{\phi}_n^0(\xi) - \hat{\phi}_n^0(\xi)}] d\xi \right|^2 \\ \leq \left(\int_B |\phi(x)|^2 dx \right) \left(\int_B |U_{\psi}(x; \kappa)|^2 dx \right) \\ \leq C \int_B |U_{\psi}(x; \kappa)|^2 dx .$$

Now, since $U_{\psi}(x; \kappa) = -\psi(x)$ on $\overset{\circ}{\Omega}$ and $\psi(x) = 0$ on $\overset{\circ}{\Omega} - \overset{\circ}{\Omega}'_h$, we have for each y in B

$$(3.40) \quad U_{\psi}(y; \kappa) = \int_{\overset{\circ}{\Omega}'_h} \psi(x) \frac{\partial G_{\kappa}(x; y)}{\partial n} dS_x$$

where S_x represents arc length along $\overset{\circ}{\Omega}$ and $G_{\kappa}(x; y)$ denotes Green's function for the operator $A - \kappa$. It thus follows from Lemma 3.2 that

$$(3.41) \quad \max_{y \in B} |U_{\psi}(y; \kappa)|^2 \leq \left(\int_{\overset{\circ}{\Omega}'_h} \left| \frac{\partial G_{\kappa}(x; y)}{\partial n} \right|^2 dS_x \right) \left(\int_{\overset{\circ}{\Omega}'_h} |\psi(x)|^2 dS_x \right)$$

$$\leq C \int_{\Omega'_h} |\psi(x)|^2 ds_x.$$

We now use the following estimates:

$$(3.42) \quad \int_{\Omega'_h} |\psi(x)|^2 ds_x \leq C \|\psi\|_{1(S)}^2 \leq C \|(-\Delta+1)\psi\|_{0(S)}^2.$$

Using Theorem 1.1, we may write (3.42) as

$$(3.43) \quad \int_{\Omega'_h} |\psi|^2 ds_x \leq C \sum_{n=1}^{\infty} \int_0^{\infty} (1 + \xi^2 + v_n)^2 |\hat{\psi}_n^0(\xi)|^2 d\xi.$$

Thus for each $\psi(x)$ in $\mathcal{J}$, we have from (3.39), (3.41) and (3.43)

$$(3.44) \quad \left| \sum_{n=1}^{\infty} \int_0^{\infty} \hat{\psi}_n^0(\xi) [\overline{\hat{\phi}_n^0(\xi) - \hat{\phi}_n(\xi; \kappa)}] d\xi \right|^2 \\ \leq C \sum_{n=1}^{\infty} \int_0^{\infty} |\hat{\psi}_n^0(\xi)|^2 (1 + \xi^2 + v_n)^2 d\xi.$$

The constant C depends only on ϕ . Set $Q_n(\xi) = \hat{\psi}_n^0(\xi)(1 + \xi^2 + v_n)$. We may express (3.44) as follows:

$$(3.45) \quad \left| \sum_{n=1}^{\infty} \int_0^{\infty} \frac{Q_n(\xi) [\overline{\hat{\phi}_n^0(\xi) - \hat{\phi}_n(\xi; \kappa)}]}{1 + \xi^2 + v_n} d\xi \right|^2$$

$$\leq C \sum_{n=1}^{\infty} \int_0^{\infty} |Q_n(\xi)|^2 d\xi .$$

It follows from Lemma 3.1 that the set of elements

$\{Q_n(\xi)\} = \{(1 + \xi^2 + v_n)\hat{\psi}_n^0(\xi)\}$ (where $\psi(x)$ may be any function in $\mathcal{G}$) is dense in H . Hence (3.45) holds for all elements in H and in particular for the elements $\{[\hat{\phi}_n^0(\xi) - \hat{\phi}_n(\xi; \kappa)] / (1 + \xi^2 + v_n)\}$. Applying (3.45) to the $\cap$ elements, we have

$$(3.46) \quad \sum_{n=1}^{\infty} \int_0^{\infty} \frac{|\hat{\phi}_n^0(\xi) - \hat{\phi}_n(\xi; \kappa)|^2}{(1 + \xi^2 + v_n)^2} d\xi \leq C .$$

The estimate (3.46) clearly implies

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{|\hat{\phi}_n(\xi; \kappa)|^2}{(1 + \xi^2 + v_n)^2} d\xi \leq C .$$

Since C is independent of κ in $\Lambda_{[a,b]}$ we may now conclude that (3.23) holds for each ϕ in $C_0^{\infty}(\cap)$. Q.E.D.

We shall next combine Lemma 3.3 with Theorem 3.3 to prove equation (3.15).

Theorem 3.4. For each ϕ in $C_0^{\infty}(\cap)$, equation (3.15) holds. That is,

$$\frac{1}{2} ((E_b + E_{b-})\phi, \phi)_{o(\Omega)} - \frac{1}{2} ((E_a + E_{a-})\phi, \phi)_{o(\Omega)}$$

$$= \sum_{a < \xi^2 + v_n < b} \int |\hat{\phi}_n(\xi)|^2 d\xi$$

where $[a, b] \in \Lambda$.

Proof: Suppose $K \in \Lambda[a, b]$ and $K = \tau + i\varepsilon$. By Theorem 3.3

$$(3.47) \quad \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \|R_{\tau-i\varepsilon} \phi\|_{L^2(\Omega)}^2 d\tau$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \sum_{n=1}^{\infty} \int_0^{\infty} \frac{|\hat{\phi}_n(\xi; K)|^2}{|\xi^2 + v_n - \bar{K}|^2} d\xi d\tau.$$

Using the properties of $\hat{\phi}_n(\xi; K)$ given by Theorem 3.1, we may interchange the order of summation and integration in (3.47). Consequently, we have

$$(3.48) \quad \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \|R_{\tau-i\varepsilon} \phi\|_{L^2(\Omega)}^2 d\tau$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{n=1}^{\infty} \int_0^{\infty} \int_a^b \frac{|\hat{\phi}_n(\xi; K)|^2}{(\xi^2 + v_n - \tau)^2 + \varepsilon^2} d\tau d\xi.$$

Combining (3.48) and (3.23) we obtain

$$\begin{aligned}
(3.49) \quad & \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \|R_{\tau-i\varepsilon} \phi\|_{L^2(\cap)}^2 d\tau \\
&= \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{\xi^2 + v_n < b+1} \int_a^b \frac{|\hat{\phi}_n(\xi; \kappa)|^2}{(\xi^2 + v_n - \tau)^2 + \varepsilon^2} d\tau d\xi \\
&= \sum_{\xi^2 + v_n < b+1} \int \lim_{\varepsilon \downarrow 0} \int_a^b \frac{|\hat{\phi}_n(\xi; \kappa)|^2}{(\xi^2 + v_n - \tau)^2 + \varepsilon^2} d\tau d\xi .
\end{aligned}$$

We now make use of the following result ([18]):

If $f(\tau, \varepsilon)$ is a continuous function of τ and ε for $0 \leq \varepsilon \leq \varepsilon_0$, and $\alpha \leq \tau \leq \beta$, then

$$(3.50) \quad \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{\alpha}^{\beta} \frac{f(\tau, \varepsilon)}{(A - \tau)^2 + \varepsilon^2} d\tau = \begin{cases} 0 & \text{if } A < \alpha \text{ or } \beta < A \\ f(A, 0) & \text{if } \alpha < A < \beta \end{cases} .$$

In our case, for each fixed ξ and n , set $f(\tau, \varepsilon) = |\hat{\phi}_n(\xi; \tau + i\varepsilon)|^2$. By Theorem 3.1, $\hat{\phi}_n(\xi; \kappa)$ is continuous in κ for fixed ξ and n . Consequently (3.50) yields:

$$\begin{aligned}
(3.51) \quad & \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \frac{|\hat{\phi}_n(\xi; \tau + i\varepsilon)|^2}{(\xi^2 + v_n - \tau)^2 + \varepsilon^2} d\tau \\
&= \begin{cases} 0 & \text{if } \xi^2 + v_n < a \text{ or } \xi^2 + v_n > b \\ |\hat{\phi}_n(\xi)|^2 & \text{if } a < \xi^2 + v_n < b \end{cases}
\end{aligned}$$

We conclude from (3.49) and (3.51)

$$(3.52) \quad \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \|R_{\tau - i\varepsilon} \phi\|_{L_2(\Omega)}^2 d\tau = \sum_{a < \xi^2 + v_n < b} \int |\hat{\phi}_n(\xi)|^2 d\xi.$$

Combining (3.52) with (3.14), we have proven the theorem.

Q.E.D.

We are now ready to prove that the generalized eigenfunctions $w_n(x; \xi)$ form a complete set.

Theorem 3.5. For each ϕ in $C_0^\infty(\Omega)$, set

$$(3.53) \quad T\phi = \{\hat{\phi}_n(\xi)\} = \left\{ \int_{\Omega} \phi(x) \overline{w_n(x; \xi)} dx \right\} = \{(\phi, w_n(\cdot; \xi))_{(\Omega)}\}.$$

Then

- (i) T can be extended to a unitary transformation from $L_2(\Omega)$ onto some subspace $\mathcal{S}$ of H .
- (ii) For each f in $L_2(\Omega)$, and each bounded upper (lower) semicontinuous function $\sigma(\lambda)$, we have

$$(3.54) \quad (\sigma(A)f)^\wedge = \{\sigma(\xi^2 + v_n) \hat{f}_n(\xi)\}.$$

Also $f \in D(A)$ if and only if $\{(\xi^2 + v_n) \hat{f}_n(\xi)\} \in H$. In this case,

$$(3.55) \quad (Af)^\wedge = \{(\xi^2 + v_n) \hat{f}_n(\xi)\}.$$

(iii) For each $f \in L_2(\Omega)$, we have

$$(3.56) \quad f = \text{l.i.m.}_{a \downarrow 0, b \uparrow \infty} \sum_{a < \xi^2 + v_n < b} \int \hat{f}_n(\xi) w_n(x; \xi) d\xi.$$

Proof: (i) Suppose $[a_0, b_0] \in \Lambda$. By Theorem 3.4 we have for each ϕ in $C_0^\infty(\Omega)$:

$$\begin{aligned} & \frac{1}{2} ((E_{b_0} + E_{b_0^-})\phi, \phi)_{(\Omega)} - \frac{1}{2} ((E_{a_0} + E_{a_0^-})\phi, \phi)_{(\Omega)} \\ &= \sum_{a_0 < \xi^2 + v_n < b_0} \int |\hat{\phi}_n(\xi)|^2 d\xi. \end{aligned}$$

Since Λ has no eigenvalues, it follows from the spectral theorem [17] that for each real number b , $E_b = E_{b^-}$. Thus we conclude that for any pair of positive numbers a, b where $b \geq a$, we have

$$\begin{aligned} (3.57) \quad & ((E_b - E_a)\phi, \phi)_{(\Omega)} = \|(E_b - E_a)\phi\|_{L_2(\Omega)}^2 \\ &= \sum_{a < \xi^2 + v_n < b} \int |\hat{\phi}_n(\xi)|^2 d\xi. \end{aligned}$$

Now the interval $[a, b]$ need not belong to Λ . Letting $b \uparrow \infty$, $a \downarrow 0$, we conclude

$$\begin{aligned}
 (3.58) \quad \|\phi\|_{L_2(\Omega)}^2 &= \int_{\Omega} |\phi(x)|^2 dx \\
 &= \sum_{n=1}^{\infty} \int_0^{\infty} |\hat{\phi}_n(\xi)|^2 d\xi = \|\hat{\phi}\|_H^2.
 \end{aligned}$$

Thus T is isometric on a dense subset of $L_2(\Omega)$. We extend it by continuity to all of $L_2(\Omega)$. Denote Tf by $\{\hat{f}_n(\xi)\}$. Equations (3.57) and (3.58) will now hold for each f in $L_2(\Omega)$. This proves part (i).

(ii) As a consequence of (3.57) and the polarization identity, we have for every pair of functions $f(x)$, $g(x)$ in $L_2(\Omega)$ (where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$):

$$\begin{aligned}
 (3.59) \quad ((E_b - E_a)f, g)_{(\Omega)} &= \sum_{a < \xi^2 + v_n < b} \int \hat{f}_n(\xi) \overline{\hat{g}_n(\xi)} d\xi \\
 &= (\chi_{[a,b]} \hat{f}, \hat{g})_H
 \end{aligned}$$

where $\chi_{[a,b]} \hat{f} = \{\chi_{[a,b]}(\xi^2 + v_n) \hat{f}_n(\xi)\}$. Let $\chi_{[a,b]}^{(\mathcal{J})} \hat{f}$ denote the orthogonal projection of $\chi_{[a,b]} \hat{f}$ onto the subspace $\mathcal{J}$. Set $\chi_{[a,b]}^{(\mathcal{J})\perp} \hat{f} = \chi_{[a,b]} \hat{f} - \chi_{[a,b]}^{(\mathcal{J})} \hat{f}$. We thus have from (3.59)

$$(3.60) \quad ((E_b - E_a)f, g)_{(\Omega)} = (\chi_{[a,b]}^{(\mathcal{J})} \hat{f}, \hat{g})_H.$$

Equation (3.58) and the polarization identity imply

$$((E_b - E_a)f, g)_{(\Omega)} = (((E_b - E_a)f)^\wedge, \hat{g})_H.$$

Combining the last equation with equation (3.60) we obtain

$$(3.61) \quad ((E_b - E_a)f)^\wedge = \chi_{[a,b]}^{(\mathcal{J})} \hat{f}.$$

It follows from (3.57) that

$$\begin{aligned} \|(E_b - E_a)f\|_{L_2(\Omega)}^2 &= \|\chi_{[a,b]} \hat{f}\|_H^2 = \\ &= \|\chi_{[a,b]}^{(\mathcal{J})} \hat{f}\|_H^2 + \|\chi_{[a,b]}^{(\mathcal{K})} \hat{f}\|_H^2. \end{aligned}$$

Equation (3.61) and the fact that T is isometric imply

$$\|(E_b - E_a)f\|_{L_2(\Omega)}^2 = \|((E_b - E_a)f)^\wedge\|_H^2 = \|\chi_{[a,b]}^{(\mathcal{J})} \hat{f}\|_H^2.$$

Comparing the last two equations, we see that $\chi_{[a,b]} \hat{f} = \chi_{[a,b]}^{(\mathcal{J})} \hat{f}$. Hence equation (3.61) gives $((E_b - E_a)f)^\wedge = \chi_{[a,b]}^{(\mathcal{J})} \hat{f}$. Part (ii) now follows using the last equation and the spectral theorem.

(iii) Suppose $f \in L_2(\Omega)$, $\psi \in C_0^\infty(\Omega)$. Equation (3.59) and the definition (3.2) of $\hat{\psi}_n(\xi)$ give

$$(3.62) \quad ((E_b - E_a)f, \psi)_{(\Omega)} = \sum_{a < \xi^2 + v_n < b} \int \hat{f}_n(\xi) \overline{\left(\int_{\Omega} \psi(x) \overline{w_n(x; \xi)} dx \right)} d\xi.$$

We may interchange the order of summation and integration on the right hand side of equation (3.62) as a consequence of Theorem (2.4)(a) and Definition (2.20). Therefore

$$\begin{aligned} (3.63) \quad & ((E_b - E_a)f, \psi)_{(\Omega)} \\ &= \int_{\Omega} \left(\sum_{a < \xi^2 + v_n < b} \int \hat{f}_n(\xi) w_n(x; \xi) d\xi \right) \overline{\psi(x)} dx \\ &= \left(\sum_{a < \xi^2 + v_n < b} \int \hat{f}_n(\xi) w_n(\cdot; \xi) d\xi, \psi \right)_{(\Omega)}. \end{aligned}$$

Since this holds for every ψ in $C_0^\infty(\Omega)$, it follows that

$$(3.64) \quad (E_b - E_a)f = \sum_{a < \xi^2 + v_n < b} \int \hat{f}_n(\xi) w_n(x; \xi) d\xi.$$

This equation immediately implies (3.56). Hence the theorem is proven. Q.E.D.

Equation (3.56) is called the completeness relation. It has not yet been proven that $\mathcal{A} = H$. This result will follow from Theorem 3.6. First we show that we may define

a continuous linear transformation T' from H into $L_2(\Omega)$ such that for each $g = \{g_n(\xi)\}$ in H , we have

$$(3.65) \quad T'g = \left(\frac{2}{\pi}\right)^{1/2} \lim_{b \uparrow \infty} \sum_{\xi^2 + v_n < b} \int g_n(\xi) w_n(x; \xi) d\xi.$$

Denote by D the set of elements $g = \{g_n(\xi)\}$ in H satisfying the conditions (1) $g_n(\xi) \equiv 0$ for $n \geq M$ where M is some positive integer depending on g , and (2) each $g_n(\xi) \in C_0^\infty(0, \infty)$. Set $L_g(f) = (g, \hat{f})_H = \sum_{n=1}^{\infty} \int_0^{\infty} g_n(\xi) \overline{\hat{f}_n(\xi)} d\xi = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} \int_0^{\infty} g_n(\xi) \left(\int_{\Omega} w_n(x; \xi) \overline{f(x)} dx \right) d\xi$, where $f \in C_0^\infty(\Omega)$, $g \in D$. It is easily seen that for each g in D , $L_g(f)$ is a continuous conjugate linear functional on $C_0^\infty(\Omega)$. Extend it to all of $L_2(\Omega)$. $L_g(f)$ now represents a bounded conjugate linear functional on $L_2(\Omega)$. Hence by the Riesz representation theorem there exists a unique function $g'(x)$ in $L_2(\Omega)$ such that

$$(3.66) \quad L_g(f) = (g', f)_{(\Omega)}.$$

It follows from this that $T'g = g'$ for each g in D , where T' is defined by (3.65). D is dense in H . It follows from (3.66) that T' is continuous on D . Hence we may extend T' to all of H . Equation (3.65) will now hold for each g in H .

Theorem 3.6. $\forall g = \{g_n(\xi)\} \in H$, we have

$$(3.67) \quad TT'g = g.$$

Proof: Set $r = \{r_n(\xi)\} = \{g_n(\xi)\} - TT'\{g_n(\xi)\}$.

We shall show that $r = 0$. Making use of the unitarity of the transformation $T_1 = T_0^{-1}$ (Theorem 1.1), it is clearly sufficient to show that

$$(3.68) \quad 0 = T_1 r = \lim_{b \uparrow \infty} \sum_{\xi^2 + v_n \leq b} \int r_n(\xi) w_n^0(\xi) d\xi.$$

We shall begin by establishing the equation

$$(3.69) \quad \sum_{\alpha < \xi^2 + v_n < \beta} \int r_n(\xi) w_n(x; \xi) d\xi = 0$$

for any numbers α, β such that $\beta \geq \alpha \geq 0$.

For any point x in Ω , set $z_m(x) = \sum_{\xi^2 + v_n \leq m} \int r_n(\xi) w_n(x; \xi) d\xi$, and

$$(3.70) \quad Z_m(x; \kappa) = \sum_{\xi^2 + v_n \leq m} \int \frac{r_n(\xi) w_n(x; \xi)}{(\xi^2 + v_n - \kappa)} d\xi$$

where $\kappa = \tau + i\varepsilon$, $\varepsilon > 0$. Since $(-\Delta - \kappa)w_n(x; \xi) = (\xi^2 + v_n - \kappa)w_n(x; \xi)$, and $w_n(x; \xi) = 0$ on $\overset{\circ}{\Omega}$, we have $(-\Delta - \kappa)Z_m = z_m$ and $Z_m = 0$ on $\overset{\circ}{\Omega}$. Obviously $z_m(x)$ and

$z_m(x; \kappa) \in L_2(\Omega)$. Therefore $z_m = (A - \kappa)^{-1} z_m$. As $m \rightarrow \infty$, $z_m \rightarrow T'r = T'g - T'TT'g = 0$ using Theorem 3.5 (c) (convergence is in the sense of $L_2(\Omega)$). We also have $\|z_m\|_{L_2(\Omega)} \leq \|(A - \kappa)^{-1}\| \cdot \|z_m\|_{L_2(\Omega)} \leq C \|z_m\|_{L_2(\Omega)}$. Thus

$$(3.71) \quad z_m \rightarrow 0 \text{ in } L_2(\Omega) \text{ as } m \rightarrow \infty.$$

Now let D be an arbitrary bounded, measurable subset of Ω . Then (3.70) and (3.71) give

$$\begin{aligned} (3.72) \quad 0 &= \lim_{m \rightarrow \infty} \int_D z_m(x; \kappa) \, dx \\ &= \lim_{m \rightarrow \infty} \int_D dx \left(\sum_{\xi^2 + v_n < m} \int \frac{w_n(x; \xi) r_n(\xi)}{\xi^2 + v_n - \kappa} \, d\xi \right) \\ &= \lim_{m \rightarrow \infty} \sum_{\xi^2 + v_n < m} \int \frac{r_n(\xi) d\xi \left(\int_D w_n(x; \xi) \, dx \right)}{\xi^2 + v_n - \kappa} \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} \frac{r_n(\xi) \, d\xi}{\xi^2 + v_n - \kappa} \int_D w_n(x; \xi) \, dx. \end{aligned}$$

Set

$$(3.73) \quad p_n^D(\xi) = \int_D \overline{w_n(x; \xi)} \, dx.$$

Clearly $\{p_n^D(\xi)\} \in H$ i.e. $\sum_{n=1}^{\infty} \int_0^{\infty} |p_n^D(\xi)|^2 d\xi < \infty$.

It follows from (3.72) and (3.73) that

$$(3.74) \quad \sum_{n=1}^{\infty} \int_0^{\infty} \frac{r_n(\xi) \overline{p_n^D(\xi)}}{\xi^2 + v_n - \kappa} d\xi = 0.$$

Replace κ by $\overline{\kappa}$ in (3.74), then subtract, integrate with respect to τ from α to β , and take the limit as $\varepsilon \downarrow 0$. Using the dominated convergence theorem, (3.11), Definition (3.73), and the fact that

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^{\infty} |r_n(\xi) \overline{p_n^D(\xi)}| d\xi &\leq \\ &\leq \left(\sum_{n=1}^{\infty} \int_0^{\infty} |r_n(\xi)|^2 d\xi \right)^{1/2} \left(\sum_{n=1}^{\infty} \int_0^{\infty} |\overline{p_n^D(\xi)}|^2 d\xi \right)^{1/2} < \infty, \end{aligned}$$

we get

$$\begin{aligned} 0 &= \lim_{\varepsilon \downarrow 0} \int_{\alpha}^{\beta} d\tau \sum_{n=1}^{\infty} \int_0^{\infty} \frac{2i\varepsilon r_n(\xi) \overline{p_n^D(\xi)} d\xi}{(\xi^2 + v_n - \tau)^2 + \varepsilon^2} \\ &= \lim_{\varepsilon \downarrow 0} 2i \sum_{n=1}^{\infty} \int_0^{\infty} r_n(\xi) \overline{p_n^D(\xi)} \int_{\alpha}^{\beta} \frac{\varepsilon d\tau d\xi}{\xi^2 + v_n - \tau)^2 + \varepsilon^2} \end{aligned}$$

$$\begin{aligned}
&= 2i \sum_{n=1}^{\infty} \int_0^{\infty} r_n(\xi) \overline{p_n^D(\xi)} \lim_{\epsilon \downarrow 0} \int_{\alpha}^{\beta} \frac{\epsilon d\tau d\xi}{(\xi^2 + v_n - \tau)^2 + \epsilon^2} \\
&= 2\pi i \int_D dx \left(\sum_{\alpha < \xi^2 + v_n < \beta} \int r_n(\xi) w_n(x; \xi) d\xi \right).
\end{aligned}$$

Since D is an arbitrary subdomain of Ω , this implies (3.69).

We now prove (3.68). Given any $\rho \geq v_1$, (3.69) implies that for $\epsilon > 0$

$$\sum_{\rho - \epsilon < \xi^2 + v_n < \rho + \epsilon} \int r_n(\xi) w_n(x; \xi) d\xi = 0.$$

Making the substitutions $\lambda = v_n + \xi^2$, this becomes

$$0 = \int_{\rho - \epsilon}^{\rho + \epsilon} \sum_{n=1}^{M_{\rho}} \frac{r_n(\sqrt{\lambda - v_n}) w_n(x; \sqrt{\lambda - v_n}) d\lambda}{2 \sqrt{\lambda - v_n}}$$

where $v_{M_{\rho}} < \rho + \epsilon < v_{M_{\rho}+1}$. Since this is true for all $\epsilon > 0$, we conclude

$$(3.75) \quad \sum_{n=1}^{N_{\rho}} \frac{r_n(\sqrt{\rho - v_n}) w_n(x; \sqrt{\rho - v_n})}{2 \sqrt{\rho - v_n}} = 0,$$

where $v_{N_{\rho}} < \rho < v_{N_{\rho}+1}$.

Now set

$$\mu(x;\rho) = \sum_{n=1}^{N_\rho} \frac{r_n(\sqrt{\rho-v_n}) \sin \sqrt{\rho-v_n} x_N \eta_n(\tilde{x})}{2 \sqrt{\rho-v_n}}$$

for $v_{N_\rho} < \rho < v_{N_\rho+1}$, $x \in S$ $\mu(x;\rho) \in C^2(S)$ and

$\mu(x;\rho) = 0$ on $\overset{\circ}{S}$. Also $-\Delta\mu = \rho\mu$. From (3.75) and the definition of $w_n(x; \sqrt{\lambda-v_n})$ it follows that for all x in Ω ,

$$\mu(x;\rho) = - \sum_{n=1}^{N_\rho} \frac{r_n(\sqrt{\rho-v_n}) v_n(x; \sqrt{\rho-v_n})}{2 \sqrt{\rho-v_n}}.$$

Thus $\mu(x;\rho)$ satisfies the radiation conditions (2.1) and (2.2) for x_N sufficiently large. Therefore, by Theorem 2.1, we have

$$(3.76) \quad 0 \equiv \mu(x;\rho) = \sum_{n=1}^{N_\rho} \frac{r_n(\sqrt{\rho-v_n}) \sin \sqrt{\rho-v_n} x_N \eta_n(\tilde{x})}{2 \sqrt{\rho-v_n}}.$$

Integrating with respect to ρ from v_{N_ρ} to $v_{N_\rho+1}$ gives

$$(3.77) \quad 0 = \int_{v_{N_\rho}}^{v_{N_\rho+1}} \sum_{n=1}^{N_\rho} \frac{\sin \sqrt{\rho-v_n} x_N \eta_n(\tilde{x}) r_n(\sqrt{\rho-v_n})}{2 \sqrt{\rho-v_n}} d\rho$$

for each positive integer N_ρ . Finally, letting $N_\rho \rightarrow \infty$, we conclude from (3.77) that for all x in S , we have

$$0 = \sum_{n=1}^{\infty} \int_{v_n}^{\infty} \frac{\sin \sqrt{\rho-v_n} x_N \eta_n(\tilde{x}) r_n(\sqrt{\rho-v_n})}{2 \sqrt{\rho-v_n}} d\rho.$$

Making the substitutions $\rho = \xi^2 + v_n$, we get

$$0 = \sum_{n=1}^{\infty} \int_0^{\infty} r_n(\xi) \sin \xi x_N \eta_n(\tilde{x}) d\xi.$$

This implies (3.68), thus proving the theorem. Q.E.D.

NOTE: Equation (3.67) is called the "orthogonality" relation, [6].

Let us now denote the functions $v_n(x; \xi)$ and $w_n(x; \xi)$ by $v_n^+(x; \xi)$ and $w_n^+(x; \xi)$, respectively. We also denote the mapping T by T^+ . Thus $T^+f = \{(f, w_n^+(\cdot; \xi))_{(\Omega)}\} = \{\hat{f}_n^+(\xi)\}$ for f in $L_2(\Omega)$. It was mentioned in Chapter 2 that in the proof of Theorem 2.2 we could have considered the functions $\mu_{\varepsilon, v}^- = (A - \lambda - v - i\varepsilon)^{-1}F$, where $v_r < \lambda < v_{r+1}$, $\varepsilon < 0$. The same proof would then have shown that there exists a function $\mu^-(x)$ in χ such that $\mu_{\varepsilon, v}^- \xrightarrow{\chi} \mu^-$ as $\varepsilon \downarrow 0$, and $\bar{A}\mu^- = \lambda\mu^-$. For x_N sufficiently large, $\mu^-(x)$ satisfies the following radiation conditions:

$$\int_{\ell}^{\infty} \mu^-(x) \eta_j(\tilde{x}) d\tilde{x} = c_j e^{-\sqrt{v_j - \lambda} x_N}$$

$$\int_{\ell_{x_N}} \frac{\partial \mu^-(x)}{\partial x_N} \eta_j(\tilde{x}) d\tilde{x} = -c_j \sqrt{v_j - \lambda} e^{-\sqrt{v_j - \lambda} x_N},$$

$j = 1, 2, \dots$. In this case for $j \leq r$, we have $\sqrt{v_j - \lambda} = i \sqrt{\lambda - v_j}$. Thus we may construct a set of incoming waves $v_n^-(x; \xi)$ which have the form:

$$v_n^-(x; \xi) = \sum_{j=1}^r c_j e^{-i \sqrt{\xi^2 + v_n - v_j} x_N} \eta_j(\tilde{x}) + \sum_{j=r+1}^{\infty} c_j e^{-\sqrt{v_j - (\xi^2 + v_n)} x_N} \eta_j(\tilde{x}),$$

where $v_r < \xi^2 + v_n < v_{r+1}$.

Set $w_n^-(x; \xi) = w_n^0(x; \xi) + v_n^-(x; \xi)$. Thus we have obtained a different set of distorted plane waves. These are expressed as the sum of a plane wave and an incoming wave. For each $\phi(x)$ in $C_0^\infty(\Omega)$, set $T^- \phi = \{\hat{\phi}_n^-(\xi)\} = \{(\phi, w_n^-(\cdot; \xi))_{(\Omega)}\}$. Theorems 3.5 and 3.6 may also be proven for the transformation T^- . The proof is exactly the same as for T^+ .

We now define two linear transformations $U^\pm$ from $L_2(\Omega)$ into the Hilbert space $H' = \bigoplus_{n=1}^{\infty} L_2(v_n, \infty; \frac{d\lambda}{2\sqrt{\lambda - v_n}})$ as follows:

For each f in $L_2(\Omega)$, set $U^\pm f = \{g_n^\pm(\lambda)\}$ where $g_n^\pm(\lambda) = \hat{f}_n^\pm(\sqrt{\lambda - v_n})$. Using the substitutions $\lambda = \xi^2 + v_n$,

it follows from Theorems 3.5 and 3.6 that U^+ and U^- are unitary transformations from $L_2(\Omega)$ onto H' . It also follows that if $f \in D(A)$, then $U^\pm(Af) = \lambda U^\pm f$. Hence U^+ and U^- each diagonalize the operator A . We conclude from this and the results of Chapter 1 that A is unitarily equivalent to A_0 .

Finally, suppose the domain Ω does not satisfy part (5) of Definition 2.1, so that A may have a countable number of point eigenvalues of finite multiplicity, [8]. We assume that Ω is still contained in S . (See Figure 2 for the two-dimensional case.)

Denote the normalized eigenfunctions of A by $w_j(x)$ and the corresponding eigenvalues by λ_j . The following expansion theorem now holds:

Theorem 3.7. For each f in $L_2(\Omega)$,

$$f = \lim_{b \uparrow \infty} \sum_{\xi^2 + v_n < b} \int \hat{f}_n(\xi) w_n(x; \xi) d\xi \\ + \lim_{J \uparrow \infty} \sum_{j=1}^J (f, w_j)_{(\Omega)} w_j(x)$$

and

$$\|f\|_{L_2(\Omega)}^2 = \sum_{n=1}^{\infty} \int_0^{\infty} |\hat{f}_n(\xi)|^2 d\xi + \sum_{j=1}^{\infty} |(f, w_j)_{(\Omega)}|^2.$$

Furthermore, $f \in D(A)$ if and only if

$$\sum_{n=1}^{\infty} \int_0^{\infty} |(\xi^2 + v_n) \hat{f}_n(\xi)|^2 d\xi + \sum_{j=1}^{\infty} |\lambda_j|^2 |(f, w_j)_{(\cap)}|^2 < \infty .$$

In this case

$$Af = \lim_{b \uparrow \infty} \sum_{\xi^2 + v_n < b} \int (\xi^2 + v_n) w_n(x; \xi) \hat{f}_n(\xi) d\xi$$

$$+ \lim_{J \uparrow \infty} \sum_{j=1}^J \lambda_j (f, w_j)_{(\cap)} w_j(x) .$$

Theorem 3.7 may be proven in essentially the same way as Theorem 3.5.

APPENDIX

Again we shall consider a perturbed semi-infinite cylinder, Ω , in R^N . In this case, however, we do not stipulate that Ω be contained in a semi-infinite cylinder. Thus, Ω satisfies conditions (1) - (4) of Definition 2.1. (See Figure 3.)

We shall show that for $\lambda > 0$ there can be at most a finite number of bounded linearly independent solutions to the boundary value problem

$$(1) \quad (\Delta + \lambda)\mu = 0 \quad \text{in } \Omega$$

$$(2) \quad \mu = 0 \quad \text{on } \overset{\circ}{\Omega}.$$

We begin with the following lemma.

Lemma. Suppose $\mu(x)$ is a real-valued function defined in Ω such that $\mu(x) \in L_2(\Omega)$ and $\mu(x)$ is a solution to boundary value problem

$$(3) \quad \Delta\mu + a_1\mu + a_2 \frac{\partial\mu}{\partial x_N} = 0 \quad \text{in } \Omega$$

$$(4) \quad \mu = 0 \quad \text{on } \overset{\circ}{\Omega},$$

where a_1, a_2 are real constants. Then there exists a constant C , independent of the function $\mu(x)$, such that

$$(5) \quad \|\mu\|_{1(\Omega)} \leq C \|\mu\|_{0(\Omega)}.$$

Proof: The divergence theorem, combined with (3) and (4), yields

$$\begin{aligned}
 \int_{\Omega} |\nabla \mu|^2 dx &= \sum_{i=1}^N \int_{\Omega_r} \left| \frac{\partial \mu}{\partial x_i} \right|^2 dx \\
 &= \int_{\Omega_r} -\mu \Delta \mu dx + \int_{\ell_r} \mu \frac{\partial \mu}{\partial x_N} d\tilde{x} \\
 &= \int_{\Omega_r} (a_1 \mu^2 + a_2 \mu \frac{\partial \mu}{\partial x_N}) dx + \int_{\ell_r} \mu \frac{\partial \mu}{\partial x_N} d\tilde{x} .
 \end{aligned}$$

Thus

$$\begin{aligned}
 (6) \quad \int_{\Omega_r} |\nabla \mu|^2 dx &- \int_{\Omega_r} \left| a_2 \mu \frac{\partial \mu}{\partial x_N} \right| dx \\
 &\leq \int_{\Omega_r} |\nabla \mu|^2 - a_2 \mu \frac{\partial \mu}{\partial x_N} dx = \int_{\Omega_r} a_1 \mu^2 dx + \int_{\ell_r} \mu \frac{\partial \mu}{\partial x_N} d\tilde{x} .
 \end{aligned}$$

Since

$$\int_{\Omega_r} \left| a_2 \mu \frac{\partial \mu}{\partial x_N} \right| dx \leq \frac{a_2^2}{2} \int |\mu|^2 dx + \frac{1}{2} \int \left| \frac{\partial \mu}{\partial x_N} \right|^2 dx,$$

we have from (6)

$$(7) \quad \frac{1}{2} \int_{\Omega_r} |\nabla \mu|^2 dx \leq \int_{\Omega_r} (a_1 + \frac{a_2^2}{2}) |\mu|^2 + \int_{\ell_r} \mu \frac{\partial \mu}{\partial x_N} d\tilde{x}.$$

Set $J_r = \int_{\ell_r} \mu \frac{\partial \mu}{\partial x_N} d\tilde{x}$. We claim that

$$(8) \quad \lim_{r \rightarrow \infty} J_r \leq 0.$$

If this were not true, then there would exist constants $r_0, C > 0$ such that $J_r \geq C$ for $r \geq r_0$. Integrating by parts, we have

$$\begin{aligned} \int_{r_0}^r J_{x_N} dx_N &= \int_{\ell} \int_{r_0}^r \mu \frac{\partial \mu}{\partial x_N} dx_N d\tilde{x} \\ &= \int_{\ell} ([\mu^2]_{r_0}^r - \int_{r_0}^r \mu \frac{\partial \mu}{\partial x_N} dx_N) d\tilde{x}. \end{aligned}$$

This implies that

$$\int_{r_0}^r J_{x_N} dx = \frac{1}{2} \int_{\ell} [\mu^2(\tilde{x}, r) - \mu^2(\tilde{x}, r_0)] d\tilde{x}.$$

Hence

$$\begin{aligned}
C(r - r_0) &\leq \int_{r_0}^r J_{x_N} dx_N \\
&= \frac{1}{2} \int_{\ell_r} \mu^2(x) d\tilde{x} - \int_{\ell_{r_0}} \mu^2(x) d\tilde{x} \\
&= C_0 + \frac{1}{2} \int_{\ell_r} \mu^2(x) d\tilde{x}.
\end{aligned}$$

From this it follows that

$$\int_{\Omega} \mu^2(x) dx = \lim_{r \rightarrow \infty} \int_{\Omega_r} \mu^2(x) dx = \infty.$$

This contradicts the fact that $\mu \in L_2(\Omega)$. Thus (8) holds.

We conclude from (7) and (8) that

$$\int_{\Omega} |\nabla \mu|^2 dx \leq 2(a_1 + \frac{a_2^2}{2}) \int_{\Omega} |\mu|^2 dx.$$

Hence the theorem is proven. Q.E.D.

Before proving the main result, we shall define a new Hilbert space, H^α . Suppose $\phi, \psi \in C_0^\infty(\Omega)$ and $\alpha > 0$.

Set

$$\langle \phi, \psi \rangle_\alpha = \int_{\Omega} \phi(x) \overline{\psi(x)} e^{-\alpha x_N} dx$$

(9)

$$\langle \phi \rangle_\alpha^2 = \int_{\Omega} |\phi(x)|^2 e^{-\alpha x_N} dx.$$

Denote by H^α the completion of $C_0^\infty(\Omega)$ under the norm defined by (9). Let S^α be the subspace of H^α consisting of all solutions of the boundary value problem (1), (2) ($\lambda > 0$), contained in H^α . Clearly, every bounded solution of (1), (2) belongs to $S^\alpha \quad \forall \alpha > 0$.

Theorem. S^α is finite-dimensional.

Proof: It is clearly sufficient to prove the result for real-valued functions defined in Ω . Thus we shall assume that all functions considered are real-valued. We first show that the unit ball in S^α is pre-compact.

Suppose $\{\mu_n\}$ represents a sequence of functions in S^α satisfying the condition that

$$(9) \quad \langle \mu_n \rangle_\alpha = \left(\int_{\Omega} \mu_n^2(x) e^{-\alpha x_N} dx \right)^{1/2} \leq 1.$$

We want to show that there exists a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ such that

$$(10) \quad \langle \mu_{n_j} - \mu_{n_i} \rangle_\alpha \rightarrow 0 \quad \text{as } i, j \rightarrow \infty.$$

Set

$$(11) \quad v_n(x) = \mu_n(x) e^{-(\alpha/2)x_N}.$$

It follows from (1) and (2) that

$$(12) \quad -\Delta v_n = \lambda v_n + \frac{\alpha^2}{4} + \alpha \frac{\partial v_n}{\partial x_N}$$

$$(13) \quad v_n = 0 \quad \text{on} \quad \overset{\circ}{\Omega}.$$

From (9) we have

$$(14) \quad \|v_n\|_{o(\Omega)} \leq 1 \quad \forall n = 1, 2, \dots.$$

(12), (13), (14), and the lemma yield

$$(15) \quad \|v_n\|_{1(\Omega)} \leq C \quad \forall n = 1, 2, \dots.$$

We now employ a Phragmen-Lindelof theorem, due to Lax [9], to conclude that there exists a $\delta > 0$ such that

$$(16) \quad \|v_n\|_{1(\Omega - \Omega_M)} \leq C e^{-\delta M},$$

C independent of n , $M = 1, 2, \dots$. For each finite subdomain Ω_M , it follows from the Rellich selection principle and Estimate (15) that there exists a subsequence $\{v_{n_j}^M\}$ such that

$$\|v_{n_j}^M - v_{n_i}^M\|_{o(\Omega_M)} \rightarrow 0$$

as $i, j \rightarrow \infty$. The sequences are chosen in such a way that $\{v_{n_j}^M\} \subseteq \{v_{n_j}^{M-1}\}$ for each M . Denote the "diagonal" subsequence $\{v_{n_j}^j\}$ by $\{v_{n_j}\}$. Clearly we have

$$\|v_{n_j} - v_{n_i}\|_{O(\Omega_M)} \rightarrow 0$$

as $i, j \rightarrow \infty$, for each $M = 1, 2, \dots$.

Now, given any $\varepsilon > 0$, choose M_0 so large that

$$(17) \quad C e^{-\delta M_0} < \frac{\varepsilon}{2}.$$

Also, choose i and j so large that

$$(18) \quad \|v_{n_j} - v_{n_i}\|_{O(\Omega_{M_0})} < \frac{\varepsilon}{2}.$$

It follows from (16), (17) and (18) that

$$(19) \quad \|v_{n_j} - v_{n_i}\|_{O(\Omega)} < \varepsilon$$

for i and j sufficiently large. From (11) and (19) we conclude that the subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ is a Cauchy sequence in H^α . We have thus shown that the unit ball in S^α is pre-compact. This clearly implies that S^α is finite-dimensional. Q.E.D.

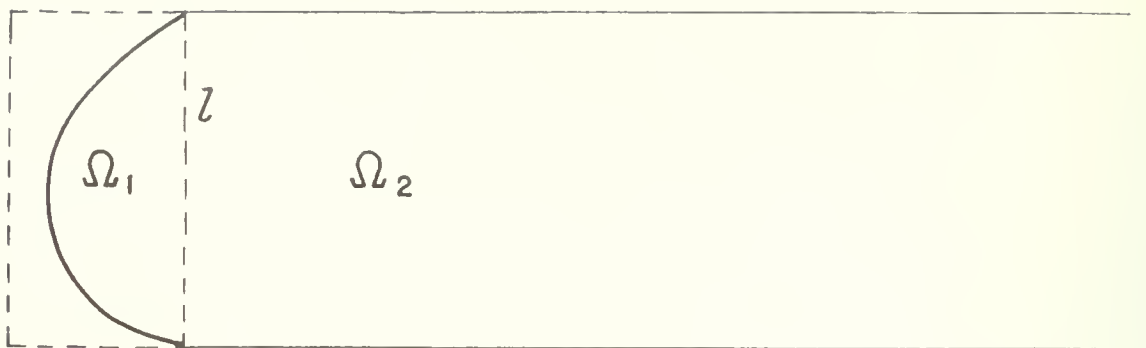


Figure 1

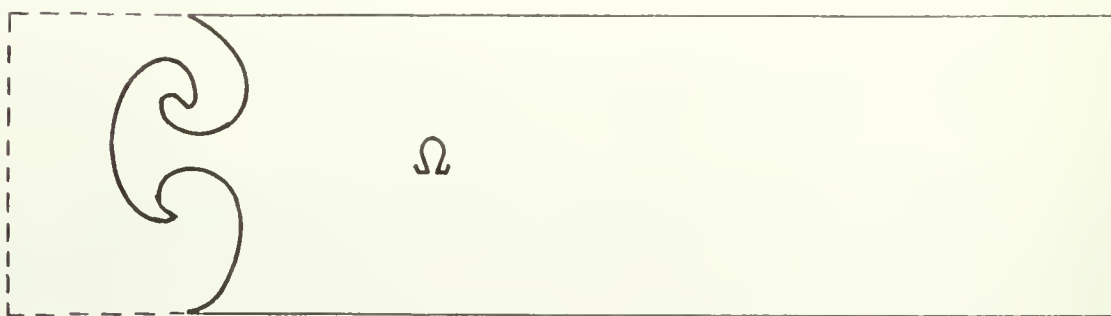


Figure 2

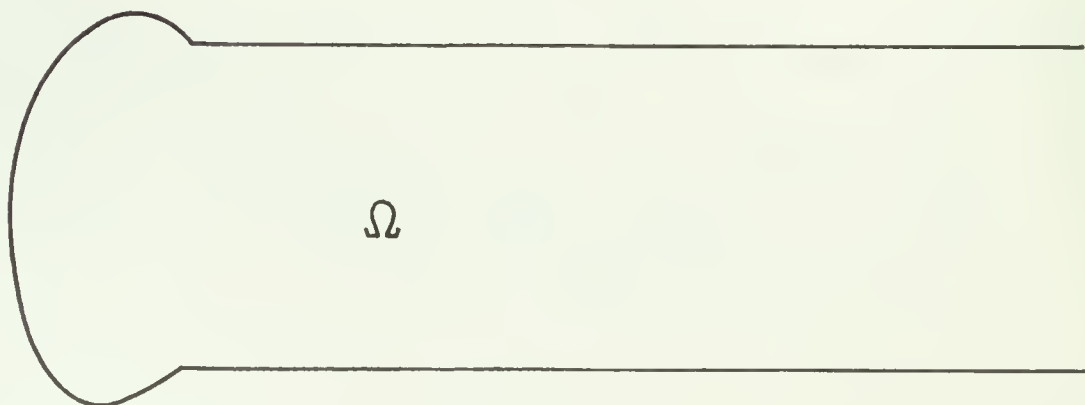


Figure 3

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